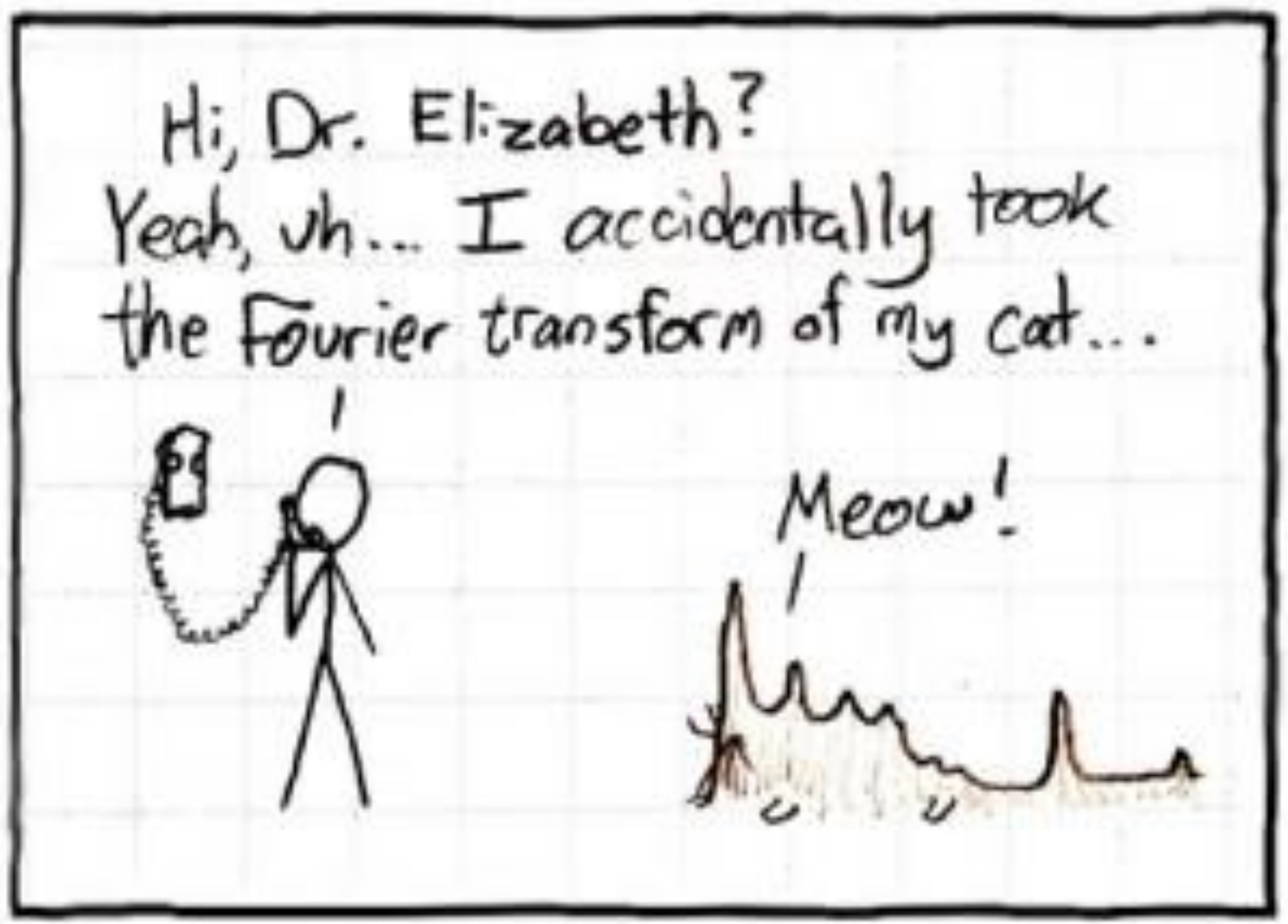
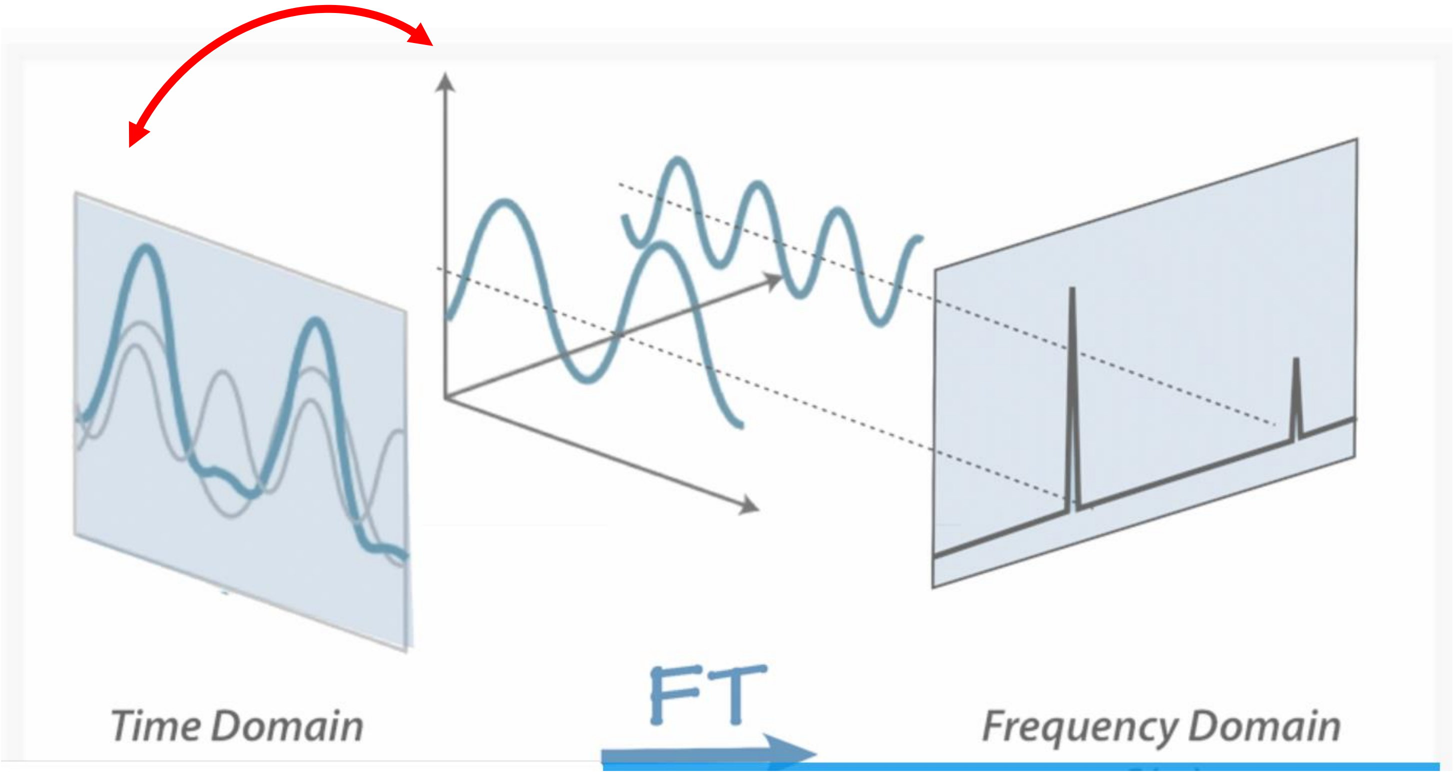



http://www.understandthefouriertransform.com/

# Introduction to Fourier optics

16TH NOVEMBER 2020

**Fourier Series** is an expansion of a periodic function in terms of an infinite sum of sines and cosines.




# Fourier Transform (FT) in temporal domain

**Fourier Transform**

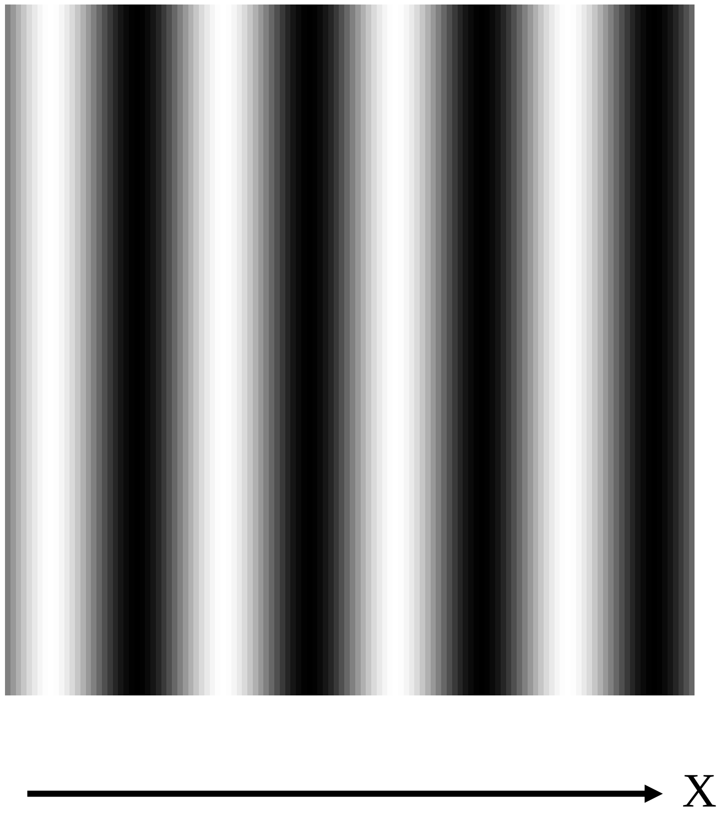
…….transforms a function in the **time domain** into **frequency domain** and the inverse is also valid….

The output in the frequency domain is expressed in terms of the (temporal) frequency.

https://aavos.eu/glossary/fourier-transform/

# Fourier Transform in spatial domain

- Now consider an image of a regular fluctuation.
- By getting a closer look, across the horizontal direction; i.e., x direction, the variation of bright and dark bands may be represented by sine or co-sinusoidal signal of **a spatial domain**.
- With an appropriate method, an image can be Fourier transformed to determine its spatial frequency components.
- For a more complicated image, a combination of harmonics are required.
- This idea is similar to the combination of harmonics to form a waveform in the temporal domain.




http://www.imagemagick.org/Usage/fourier/

# Fourier-transform pair : spatial position *x* and angular spatial frequency *k*

- Since **an image** or **optical information** under investigation is spatial distributed, the Fourier transform pair involves **spatial position *x* and angular spatial frequency *k***.
- Fourier-transform pair in **one dimension** can be written as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k)\, e^{-ikx} dk$$

$$F(k) = \int_{-\infty}^{\infty} f(x)\, e^{ikx} dx$$

- $F(k)$ is the Fourier transform of $f(x)$.

# Problem 0 : slit function

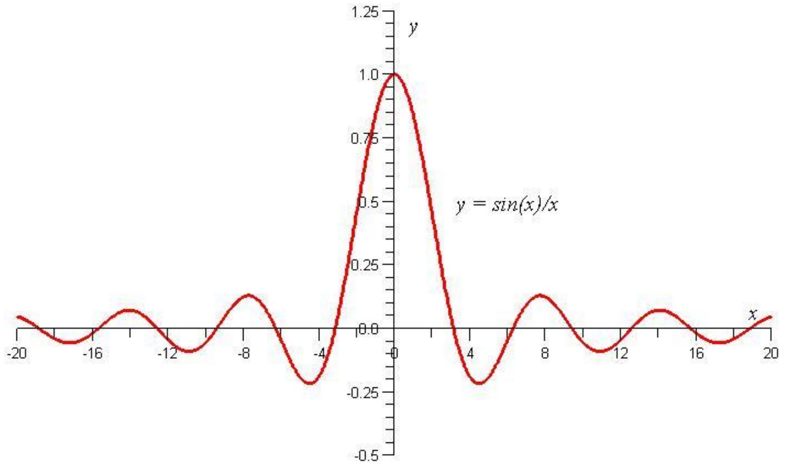
- Given a slit function in spatial domain as: $f(x) = \begin{cases} 1 & ; |x| < \frac{a}{2} \\ 0 & |x| > \frac{a}{2} \end{cases}$

- Determine the Fourier transform of $f(x)$ in the spatial frequency domain

# Solution

- Recall $F(k) = \int_{-\infty}^{\infty} f(x)\, e^{ikx} dx$
- Substituting *f*(*x*) into the Fourier transform, we have $F(k) = \int_{-b/2}^{b/2} e^{ikx} dx$
- $F(k) = \frac{1}{ik}\left(e^{ik\frac{b}{2}} - e^{-ik\frac{b}{2}}\right) = \frac{2}{k} sin\,\frac{kb}{2} = b\,\frac{\sin\frac{kb}{2}}{\frac{kb}{2}}$=……………………

Example of sinx/x function graph

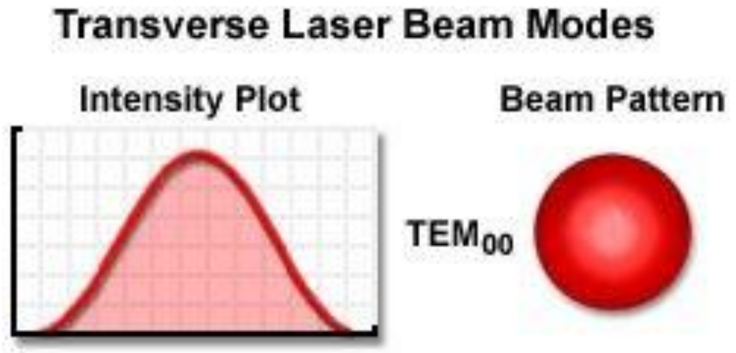


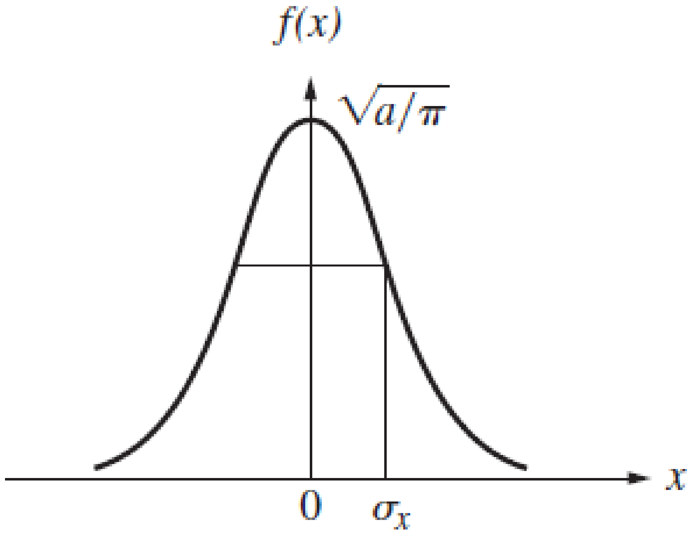
# Problem 1
# Transform a Gaussian function

- Evaluate the Fourier transform of the Gaussian probability function,

$$f(x) = Ce^{-ax^2}; \text{ where } C = \sqrt{\frac{a}{\pi}}$$

- An example of the bell-shaped curve is the cross-sectional irradiance distribution of a laser beam in the $TEM_{00}$ mode.



# Solution

- Recall the Fourier transform $F(k) = F\{f(x)\}$

$$F(k) = \int_{-\infty}^{\infty} f(x)\, e^{ikx} dx = \int_{-\infty}^{\infty} \left(Ce^{-ax^2}\right) e^{ikx} dx$$

$$= \int_{-\infty}^{\infty} \left(Ce^{-ax^2+ikx}\right) e^{k^2/4a} e^{-k^2/4a}\, dx$$

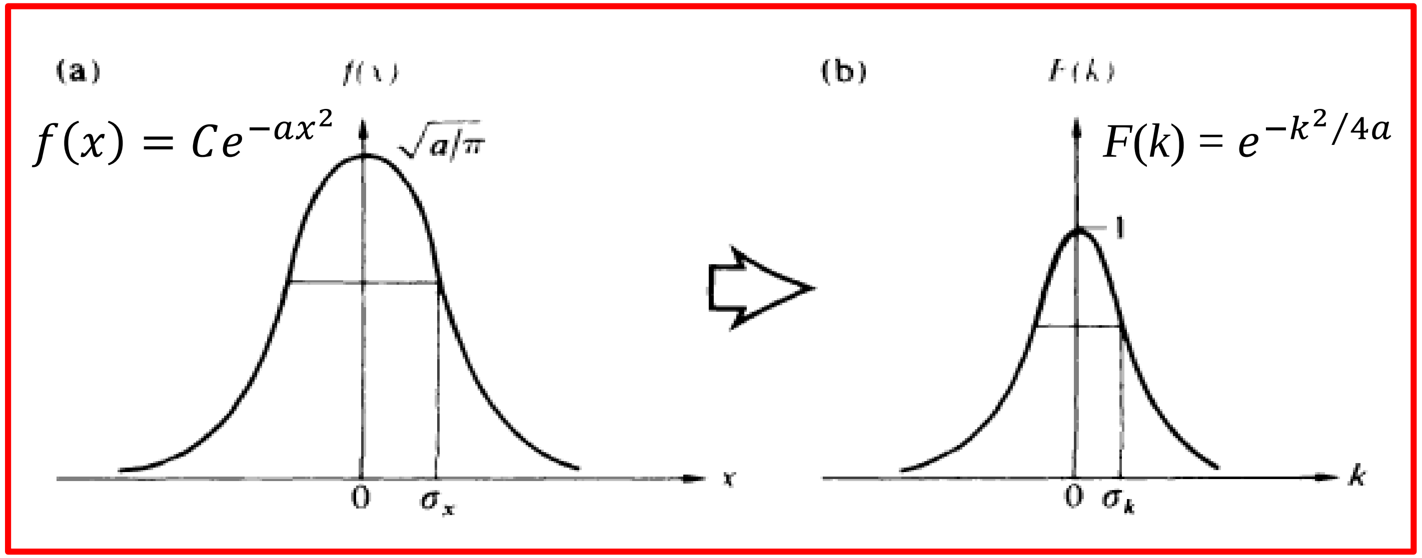
$$= \int_{-\infty}^{\infty} \left(Ce^{-\left(x\sqrt{a}-ik/2\sqrt{a}\right)^2 - k^2/4a}\right) dx$$

Letting $x\sqrt{a} - i\, k/2\sqrt{a} = \beta$ yields $F(k) = \frac{C}{\sqrt{a}} e^{-k^2/4a} \int_{-\infty}^{\infty} e^{-\beta^2}\, d\beta$

$$= e^{-k^2/4a} \quad ; \because \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

# Solution (cont.)

- Therefore, $F(k) = F\{f(x)\} = e^{-k^2/4a}$; still in the form of Gaussian function with *k* as the variable.



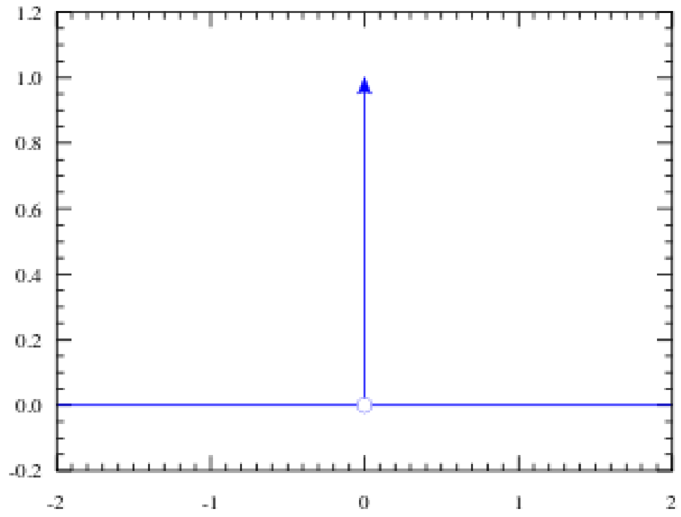

What are $\sigma_x$ and $\sigma_k$?

- This is evident that as *a* increases, *f*(*x*) becomes narrower while, in contrast, *F*(*k*) broadens.
- In other words, the shorter the pulse length, the broader the spatial frequency bandwidth.

# The Dirac Delta Function

- In the space domain, an extremely small bright source of light embedded in a dark background is highly localized, two-dimensional, spatial pulse-a spike irradiance.
- A convenient idealized mathematical representation of this sharply peaked stimulus is the **Dirac delta function (or Dirac impulse)** $\delta(x)$.

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(x)\, dx = 1$$




- This is an infinitely narrow pulse on $x = 0$. It is also known as the **unit impulse function**.

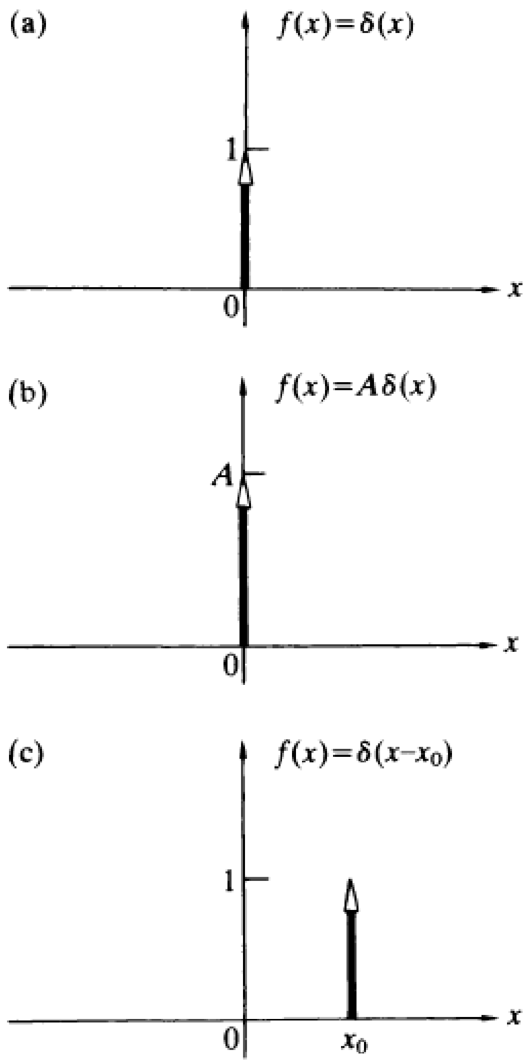
# The sifting property of δ function

- Generally, with the shift of origin of an amount $x_0$, the delta function can be written as

$$\delta(x - x_0) = \begin{cases} 0 & x \neq x_0 \\ \infty & x = x_0 \end{cases}$$

- This leads to a general form of the **sifting property** of the delta function,

$$\int_{-\infty}^{+\infty} \delta(x - x_0) f(x)\, dx = f(x_0)$$



# Problem 2
# Fourier transform of δ function

1) Determine the Fourier transform of $\delta(x - x_0)$

2) Determine the Fourier transform of $\sum_j \delta(x - x_j)$

3) If $x_1 = +d/2$ and $x_2 = -d/2$, determine the Fourier transform of $\delta(x - x_1) + \delta(x - x_2)$

# Solution

1) Recall $F(k) = \int_{-\infty}^{\infty} f(x)\, e^{ikx} dx$

By substituting $f(x) = \delta(x - x_0)$, $F(k) = \int_{-\infty}^{\infty} \delta(x - x_0)\, e^{ikx} dx$ .

Applying the sifting property, $F(k) = \int_{-\infty}^{\infty} \delta(x - x_0)\, e^{ikx} dx = e^{ikx_0}$ .

2) Again, $F(k) = \int_{-\infty}^{\infty} f(x)\, e^{ikx} dx$

By substituting $f(x) = \sum_j \delta\left(x - x_j\right)$ and applying the sifting property.

We have $F(k) = \sum_j \int_{-\infty}^{+\infty} \delta\left(x - x_j\right) e^{ikx} dx = \sum_j e^{ikx_j}$

If the function can be written as a sum of individual functions, its transform is simply the sum of the transform of the component functions.

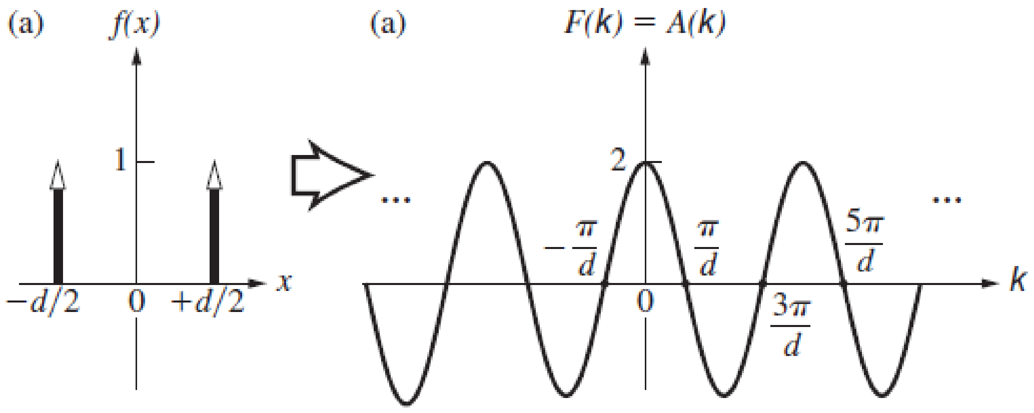
If $j \rightarrow \infty$, this summation represents a **comb function**.
In optics it is used to describe periodic structures such as diffraction gratings.

# Solution (cont.)

3) Using the result from 2), Fourier transform of $\delta(x - x_1) + \delta(x - x_2)$ can be written as

$$F(k) = e^{ikx_1} + e^{ikx_2} = e^{ik\frac{d}{2}} + e^{-ik\frac{d}{2}} = 2\cos\frac{kd}{2}$$

Thus the transform of the sum of these two symmetrical δ-function is a cosine function and vice versa.




Two delta functions and their cosine-function transform

https://www.cv.nrao.edu/course/astr534/FourierTransforms.html

# Problem 3
# Fourier transform of asymmetric function

**Calculate the Fourier transform of *f(x)***

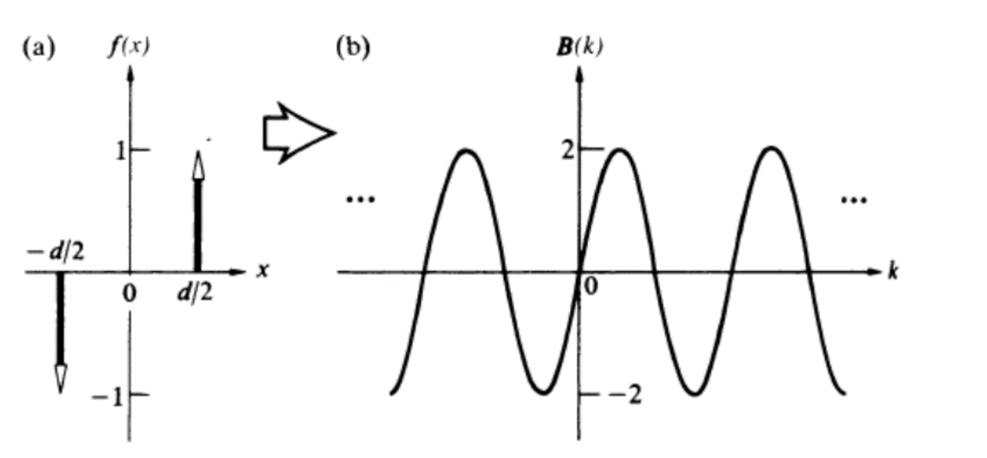
$$f(x) = \delta\left[x - \left(+\frac{d}{2}\right)\right] - \delta\left[x - \left(-\frac{d}{2}\right)\right]$$

# Solution

- Recall $F(k) = \int_{-\infty}^{\infty} f(x)\, e^{ikx} dx$
- Substituting $f(x) =$ into the above equation, then

$$F(k) = \int_{-\infty}^{\infty} \{\delta\left[x - \left(+\frac{d}{2}\right)\right] - \delta\left[x - \left(-\frac{d}{2}\right)\right] e^{ikx}\} dx;$$ **using the sifting property**

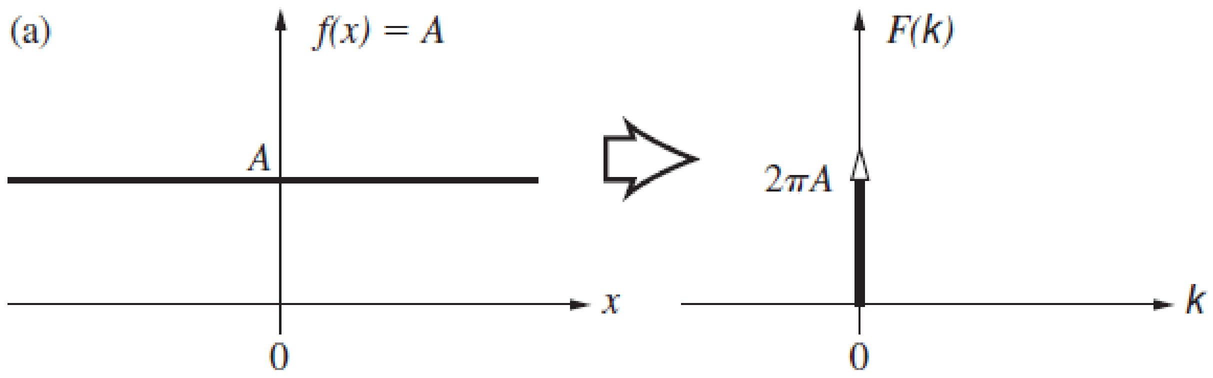
$$= e^{ik\frac{d}{2}} - e^{-ik\frac{d}{2}} = 2i\sin kd/2$$




Two delta functions and their real sine-function transform

# Problem 4

- Show that the Fourier transform of a constant A is the delta function with an amplitude 2πA.



# Solution

- Recall $F(k) = \int_{-\infty}^{\infty} f(x)\, e^{ikx} dx$
- Substituting $f(x) = A$ into the above equation: $F(k) = \int_{-\infty}^{\infty} A\, e^{ikx} dx$

$$= A\int_{-\infty}^{\infty} e^{ikx}\, dx \qquad \text{------(A)}$$

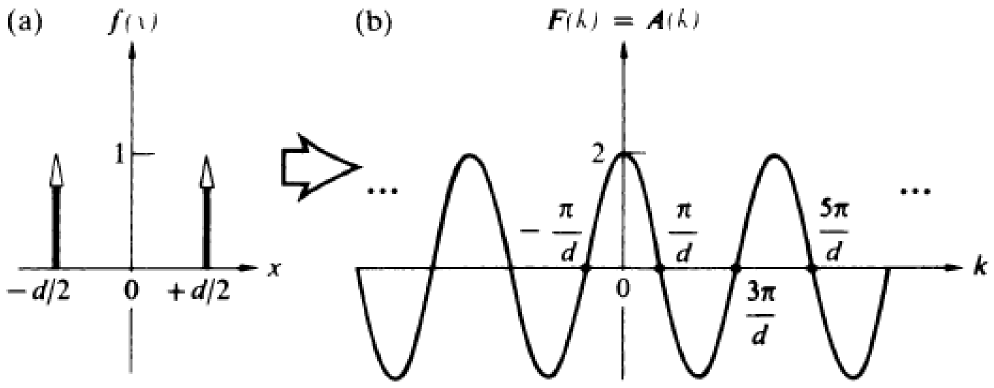
- We have to evaluate $\int_{-\infty}^{\infty} e^{ikx}\, dx$ and this can be done indirectly by using the fact that

$$F^{-1}(\delta(k)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(k)\, e^{-ikx} dk = \frac{1}{2\pi} \text{ ; from sifting property}$$

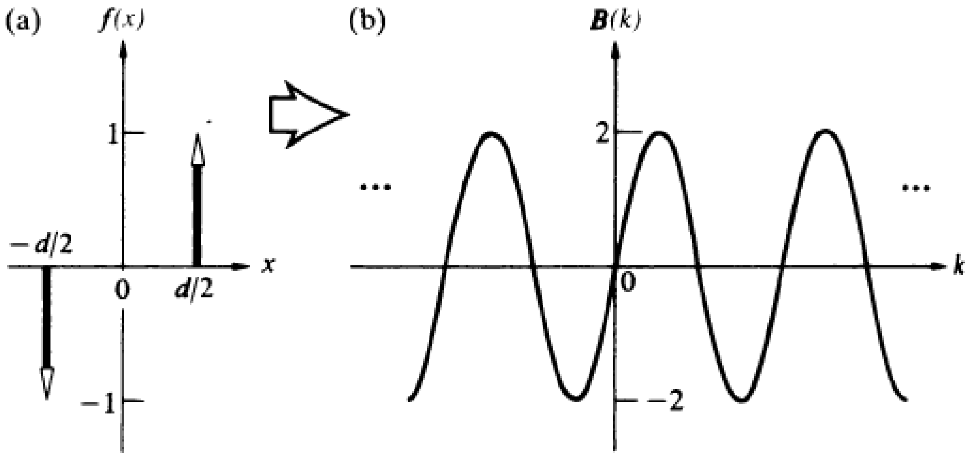
- We then have $FF^{-1}(\delta(k)) = \delta(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\, dx$ ----------(B)
- Substitute (B) into (A), $F(k) = 2\pi A\, \delta(k)$

# Fourier transform of two symmetrical and asymmetrical δ functions

**Two delta functions and their cosine-function transform**




**Two delta functions and their sine-function transform**




- This shows that the Fourier transform of two symmetrical delta functions gives a cosine function.
- Also the Fourier transform of real and even function will also be real and even.

# Two dimensional transform

- Optics generally involved two-dimensional signal: for example, the field across an aperture or the flux-density distribution over and image plane.
- The Fourier transform pair take the form,

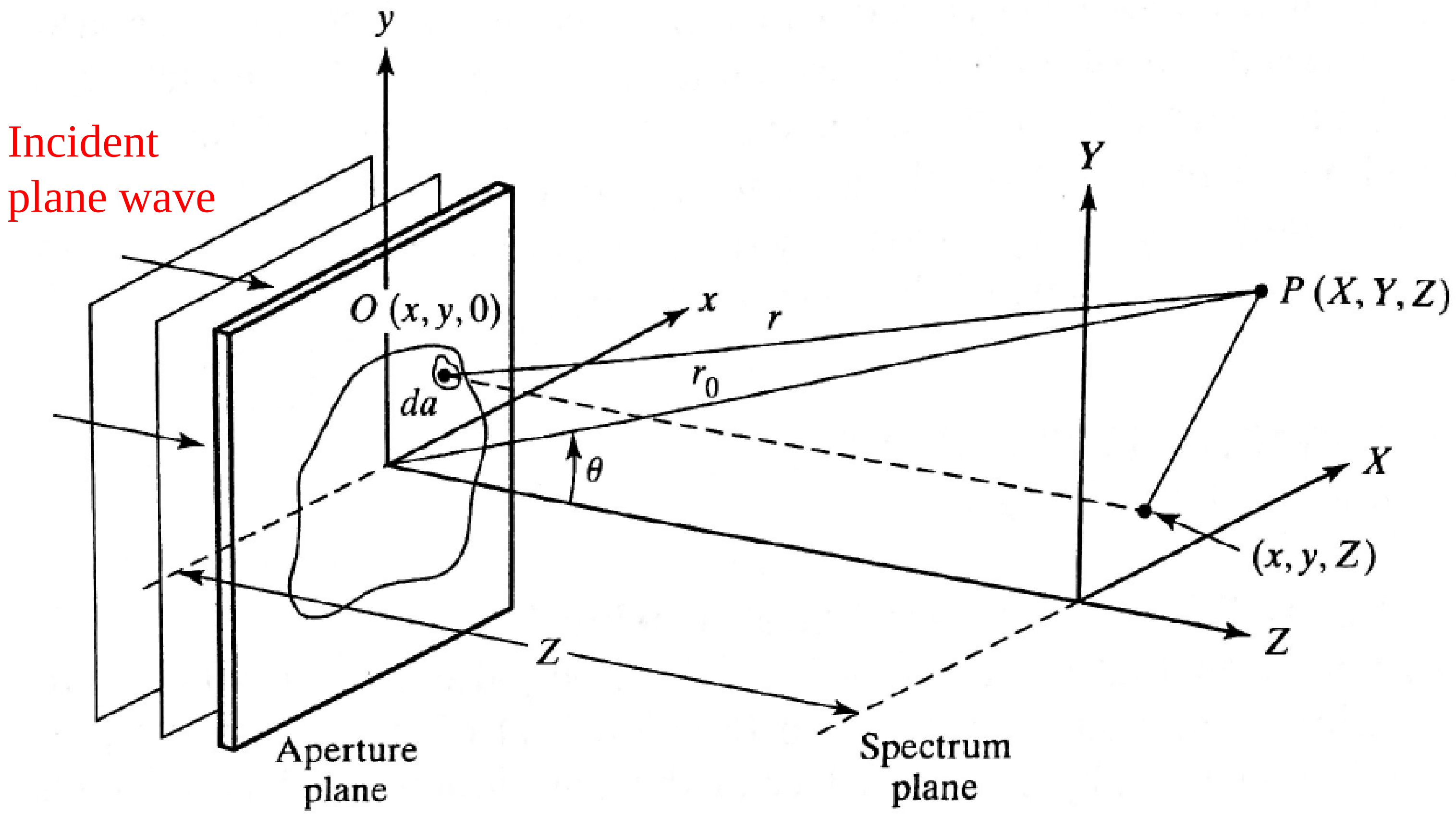
$$f(x,y)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int F(k_x,k_y)e^{-i(k_x x+k_y y)}dk_x dk_y$$

Any non periodic function of two variables $f(x,y)$ can be synthesized from a distribution of plane waves, each with amplitude $F(k_x,k_y)$ and constant phase.

and $$\mathrm{F}(k_x,k_y)=\int_{-\infty}^{\infty}\int f(x,y)e^{i(k_x x+k_y y)}dxdy$$

- Where $k_x$ and $k_y$ are **angular spatial frequencies**.

# Fraunhofer diffraction (1)




**Fraunhofer diffraction in the spectrum XY-plane due to an aperture in the xy-plane.**

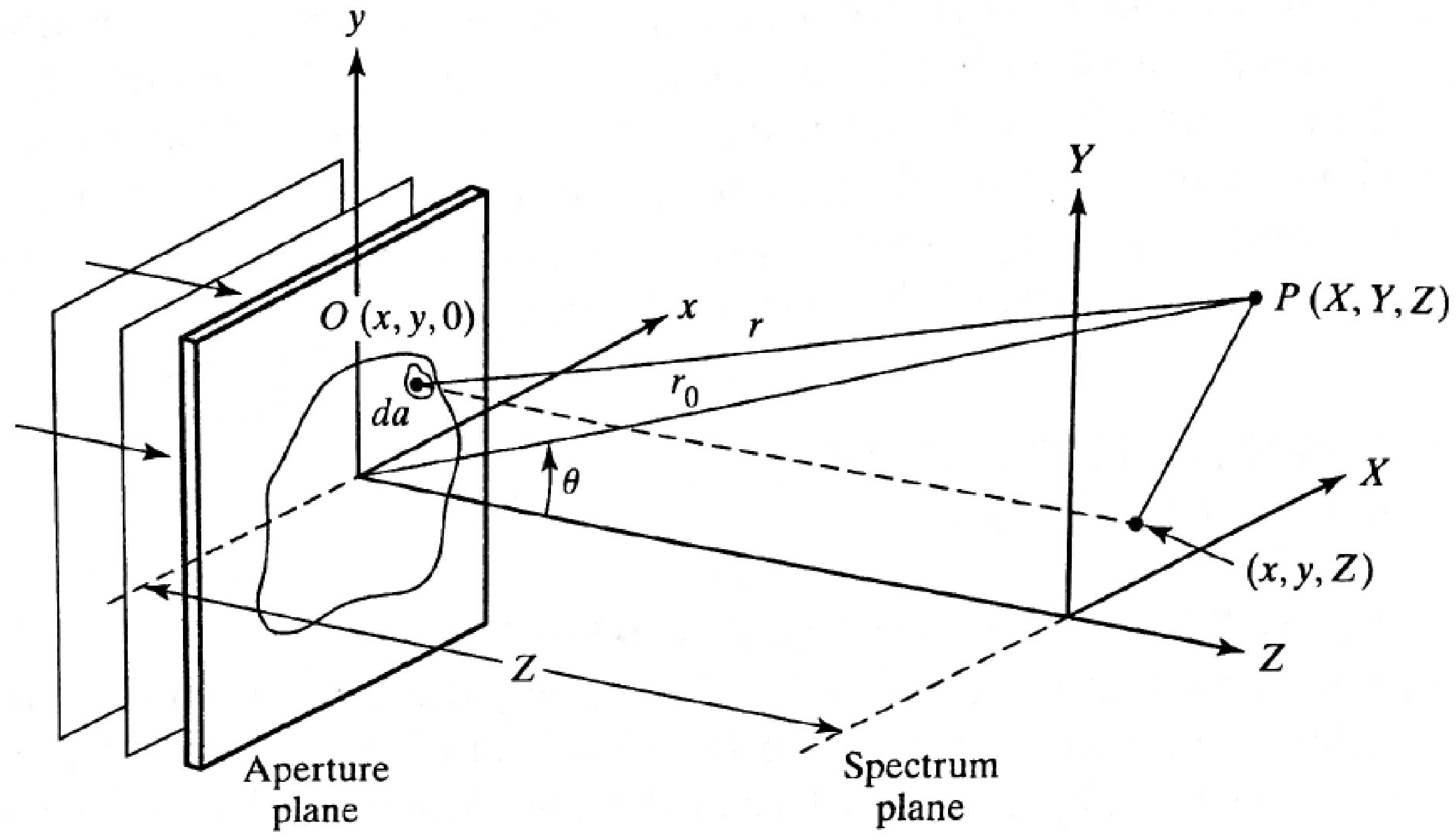
- Consider the Fraunhofer diffraction pattern due to an arbitrary aperture situated in an xy-plane (**Aperture plane**).
- Plane monochromatic waves diffract from the aperture (xy) plane.
- The diffraction pattern is observed in the XY-plane, called "**spectrum plane**", a distance Z along the axis.
- The contribution $dE_p$ at an arbitrary point P due to the light amplitude from an elemental area *da* surrounding point O in aperture is given by

$$dE_p = \left(\frac{E_A da}{r}\right) e^{i(\omega t - kr)}$$

# Fruaunhofer diffraction (2)

- Recall the contribution of electric field $dE_p$ at point P on spectrum plane: $dE_p = \left(\frac{E_A da}{r}\right) e^{i(\omega t - kr)}$
- **Amplitude of the contribution term decreases with distance *r*** (distance from point O to point P).
- The illumination of the aperture may be non-uniform and generally given as $E_A = E_A(x,y)$.
- According to the figure in the previous page, $r$ can be approximately given as $r = r_0\left[1 - \frac{(xX+yY)}{r_0^2}\right]$ (**derived in problem 5**)
- By substituting $r$ in the phase of $dE_p$ and $r$ in amplitude with Z, $dE_p = \left(\frac{E_A dxdy}{Z}\right) e^{i\omega t} e^{-ik\left[r_0 - \frac{(xX+yY)}{r_0}\right]}$
- Upon integration over the area of the aperture, thus $E_p = \left(\frac{e^{i(\omega t - kr_0)}}{Z}\right) \iint E_A(x,y) e^{ik\frac{(xX+yY)}{r_0}} dxdy$

# Problem 5




- From the figure,

$r^2 = (X - x)^2 + (Y - y)^2 + (Z - 0)^2$

and

$r_0^2 = X^2 + Y^2 + Z^2$

so that $r^2 = r_0^2 - 2xX - 2yY + (x^2 + y^2)$

$\therefore r = r_0 \left[1 - 2\frac{(xX + yY)}{r_0^2}\right]^{\frac{1}{2}}$ ; $\because x, y$ negligible

Using binomial expansion $(1 + u)^{\frac{1}{2}}$

$= 1 + \left(\frac{1}{2}\right)u + \cdots$

$\therefore r = r_0 \left[1 - \frac{(xX + yY)}{r_0^2}\right]$

# Fraunhofer diffraction (3)

- Define the **relative amplitude distribution** $A_p$ of the electric field in the spectrum plane.

$$A_p = ZE_p e^{-i(\omega t - kr_0)} = \iint E_A(x,y) e^{ik\frac{(xX+yY)}{r_0}} dxdy$$

- Also, introduction the **angular spatial frequencies**, $k_X \equiv \frac{kX}{r_0}$ and $k_Y \equiv \frac{kY}{r_0}$

- This gives

$$A_p(k_X, k_Y) = \iint E_A(x,y) e^{i(k_X x + k_Y y)} dxdy$$

- This shows that the **amplitude distribution** or **Fraunhofer diffraction pattern** $A_p\ (k_X, k_Y)$ actually is the **2D Fourier transform** of the **aperture function** $E_A\ (x,y)$.
- This also reveals that the **inverse transform** gives

$$E_A(x,y) = \frac{1}{(2\pi)^2} \iint A_p(k_X, k_Y) e^{-i(k_X x + k_Y y)} dk_X dk_Y$$

# Fourier method in diffraction theory

- We have arrived at the key point: **the field distribution in the Fraunhofer diffraction pattern is the Fourier transform of the field distribution across the aperture (i.e., the aperture function)**

$$A_p(k_X, k_Y) = \iint E_A(x, y) e^{i(k_X x + k_Y y)} dx dy \quad \text{or} \quad A_p(k_X, k_Y) = F\big(E_A(x, y)\big)$$

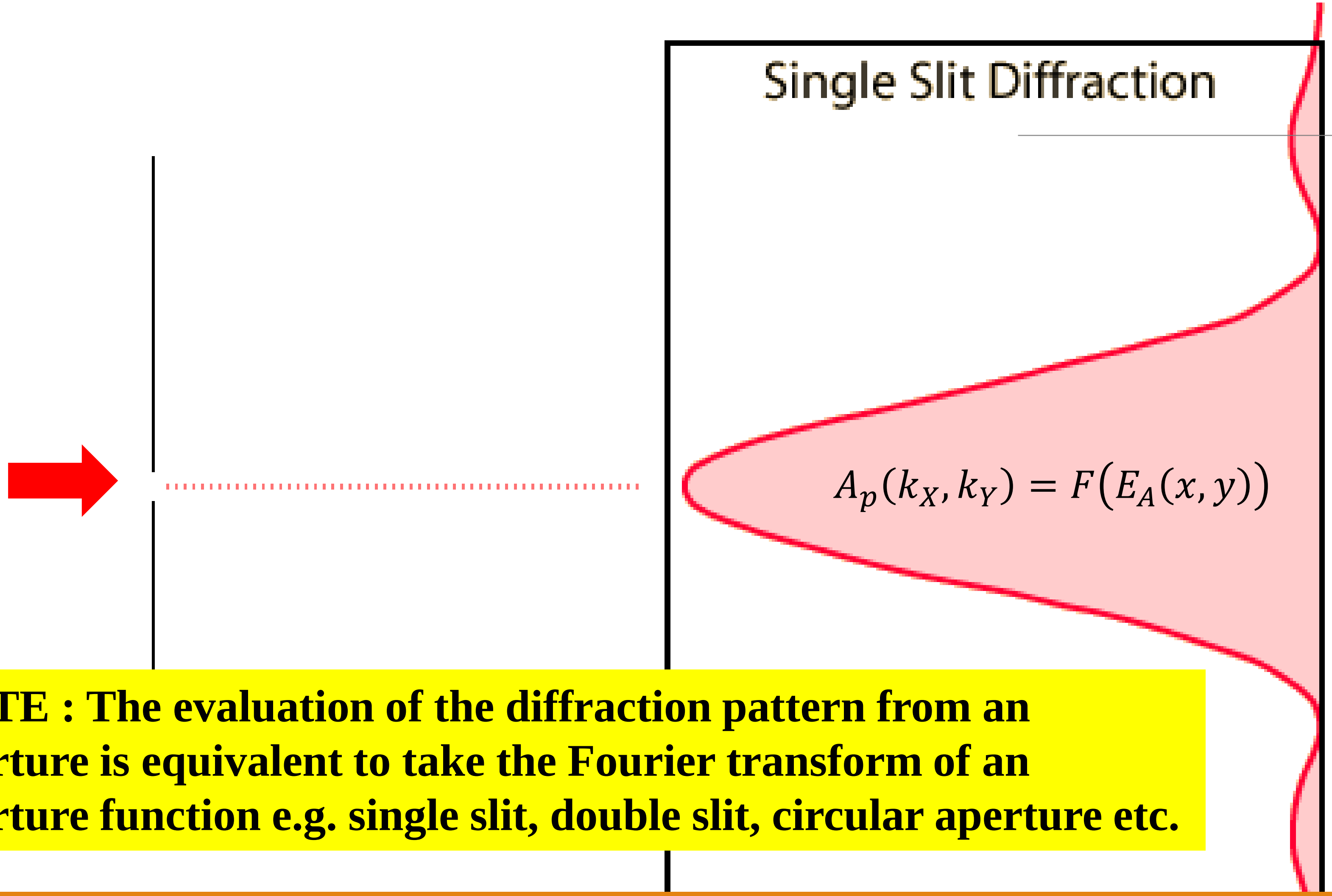
- **For each point on the image plane (spectrum plane), there is a corresponding spatial frequency.**

- The inverse transform is then

$$E_A(x, y) = \frac{1}{(2\pi)^2} \iint A_p(k_X, k_Y) e^{-i(k_X x + k_Y y)} dk_X dk_Y \quad \text{or} \quad E_A(x, y) = F^{-1}\{A_p(k_X, k_Y)\}$$

# Visual concept of the diffraction

**Field distribution across an aperture described by the aperture function $E_A\,(x, y)$**




**Fraunhofer diffraction pattern = Fourier transform of the aperture function**

**NOTE : The evaluation of the diffraction pattern from an aperture is equivalent to take the Fourier transform of an aperture function e.g. single slit, double slit, circular aperture etc.**

https://ef.engr.utk.edu/hyperphysics/hbase/phyopt/mulslid.html

# Problem 6
# Fourier transform of a Single slit

- Assuming that there are no phase or amplitude variations across the aperture, **the aperture function $E_A$(x,y)** has the form of a square pulse,

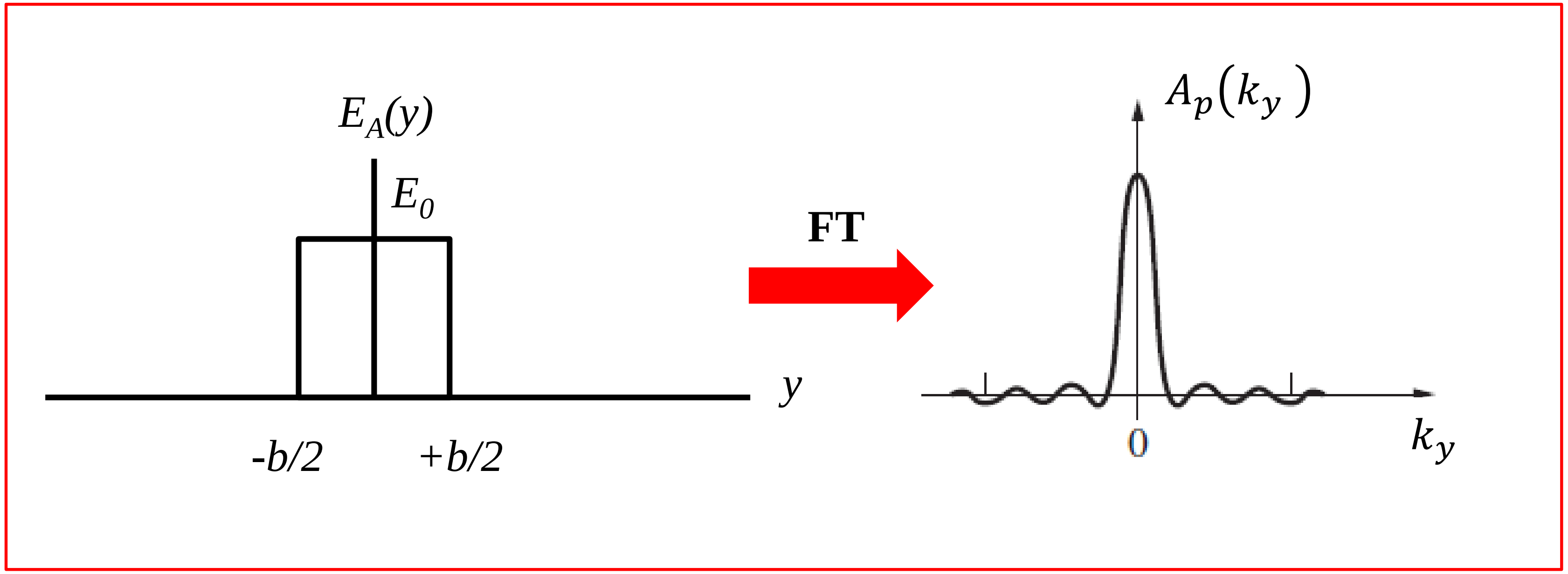
$$E_A(x,y) = \begin{cases} E_0 & ; |y| \le \dfrac{b}{2} \\ 0 & ; |y| > \dfrac{b}{2} \end{cases}$$

- Determine the field distribution from the single slit on the spectrum plane.

# Solution

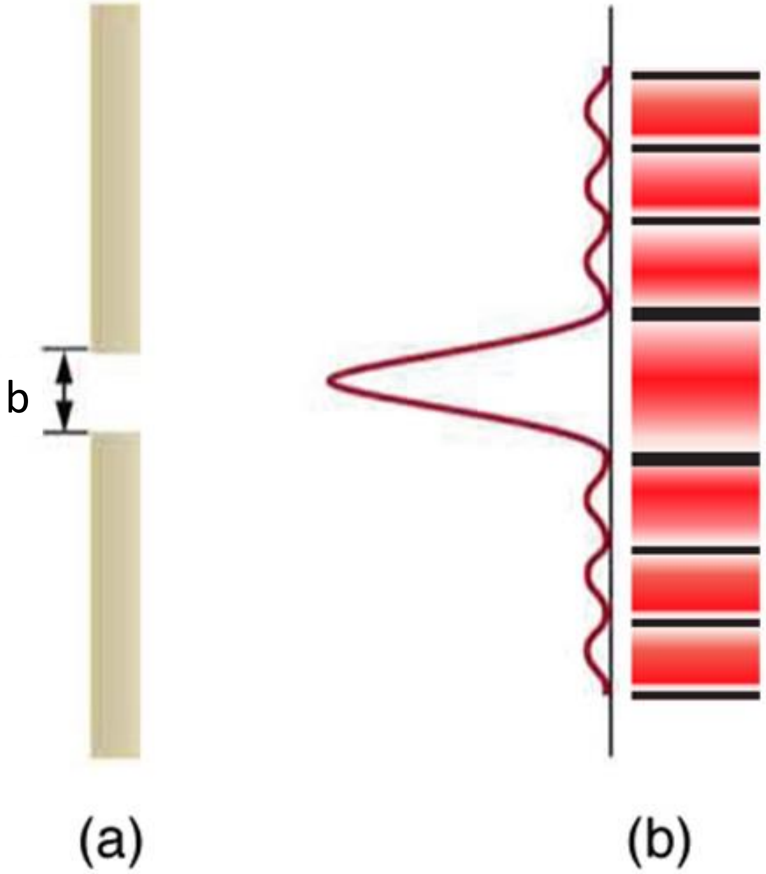
- From the given aperture function, this problem is really 1D Fourier transform along y axis.

$$A_p(k_y) = F(E_A(y))$$
$$= \int_{-\infty}^{\infty} E_A(y)\, e^{iky} dy$$
$$= E_0 \int_{-\frac{b}{2}}^{\frac{b}{2}} e^{ik_Y y} dy$$
$$= E_0 b \sin c\left(\frac{k_Y b}{2}\right)$$




- **Note that the Fourier transform gives the diffraction in terms of the electric field distribution NOT the irradiance.**
- **The diffraction pattern is composed a large number of spatial frequencies.**

# Single slit Fraunhofer irradance diffraction pattern




(a) Single slit diffraction pattern. Monochromatic light passing through a single slit has a central maximum and many smaller and dimmer maxima on either side. The central maximum is six times higher than shown.

(b) The drawing shows the bright central maximum and dimmer and thinner maxima on either side.

$$I = A^2{}_p(k_y\,) = E^2{}_0 b^2 \sin^2 c\left(\frac{k_Y b}{2}\right)$$

https://openstax.org/books/college-physics/pages/27-5-single-slit-diffraction

# Problem 7
# Fourier transform of a rectangular aperture

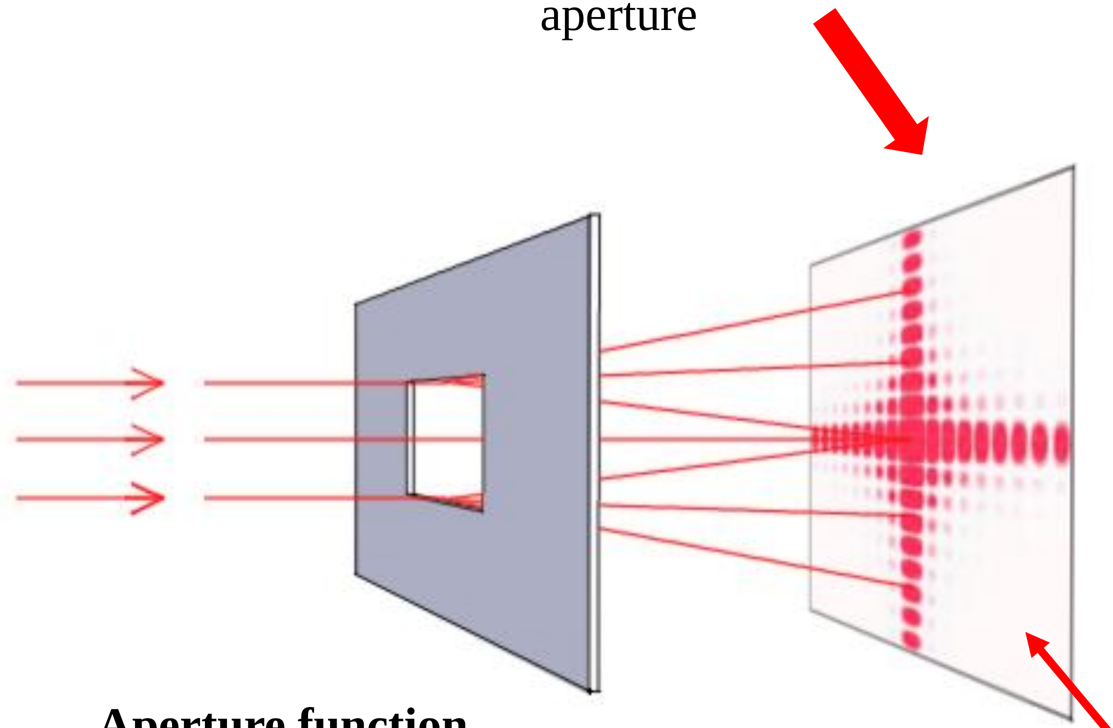
- Assuming that there are no phase or amplitude variations across the aperture, **the aperture function $E_A(x,y)$** is given by,

$$E_A(x,y) = \begin{cases} E_0 & ; |x| \leq \frac{a}{2}, |y| \leq \frac{b}{2} \\ 0 & ; |x| > \frac{b}{2}, |y| > \frac{b}{2} \end{cases}$$

- Determine the field distribution from the rectangular aperture on the spectrum plane.

# Solution

Diffraction pattern or Fourier transform of the rectangular aperture



$$A_p(k_X, k_Y) = F\left(E_A(x,y)\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_A(x,y)\, e^{i(k_X x + k_Y y)} dxdy$$

$$= E_0 \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{ik_X x} dx \int_{-\frac{b}{2}}^{\frac{b}{2}} e^{ik_Y y} dy$$

$$= E_0 ab \sin c\left(\frac{k_X a}{2}\right) \sin c\left(\frac{k_Y b}{2}\right)$$

**Aperture function**

$$E_A(x,y) = \begin{cases} E_0 & ; |x| \le \frac{a}{2}, |y| \le \frac{b}{2} \\ 0 & ; |x| > \frac{b}{2}, |y| > \frac{b}{2} \end{cases}$$

**A large number of spatial frequencies are distributed over the spectrum surface.**

# Diffraction pattern of a grating

**Fraunhofer diffraction pattern on the spectrum plane**

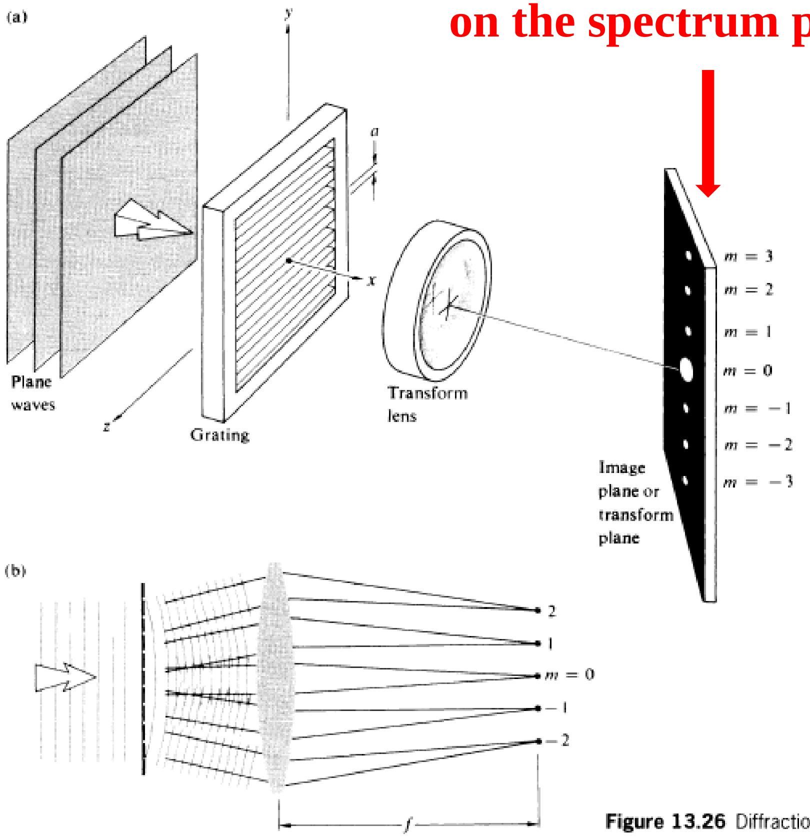
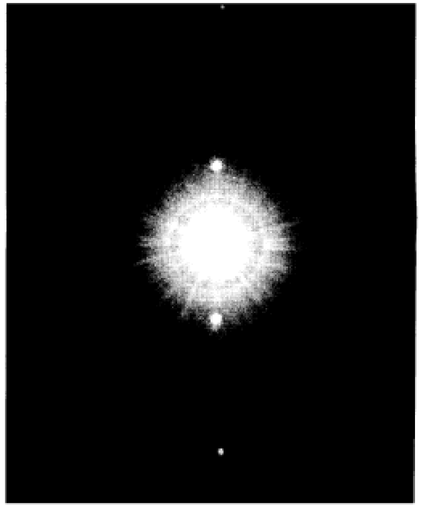
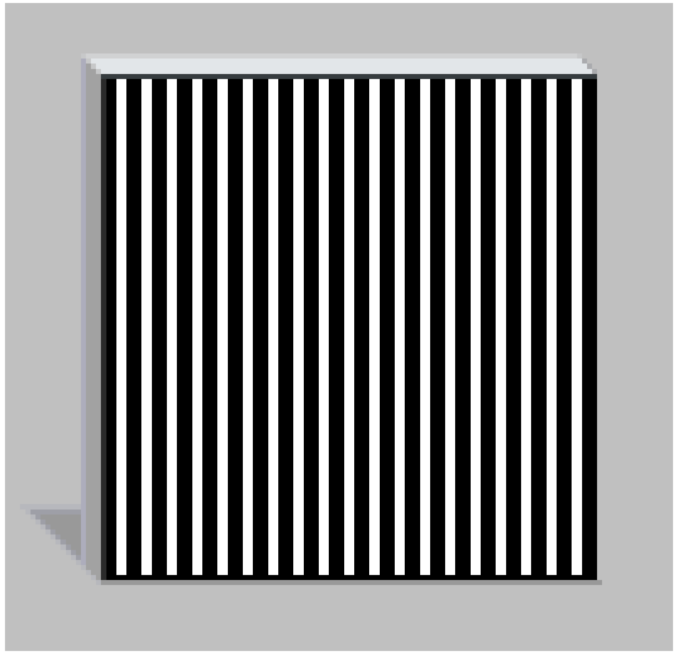



**Figure 13.26** Diffraction pattern of a grating.



- For a **transmission grating**, the concept of the Fourier transform can be applied in creating the diffraction pattern.
- The **aperture function** illuminated by a plane wave is considered to be a periodic step function.
- The Fourier transform lens helps to shorten the distance to the image plane.
- The Fraunhofer diffraction pattern, which is Fourier transform, is produced on the image plane or spectrum plane.
- **Diffraction spots on the image plane represent the spatial frequencies.**
- **As the spots in the image plane get farther from the central axis, their associated spatial frequencies increase.**

# Ronchi ruling



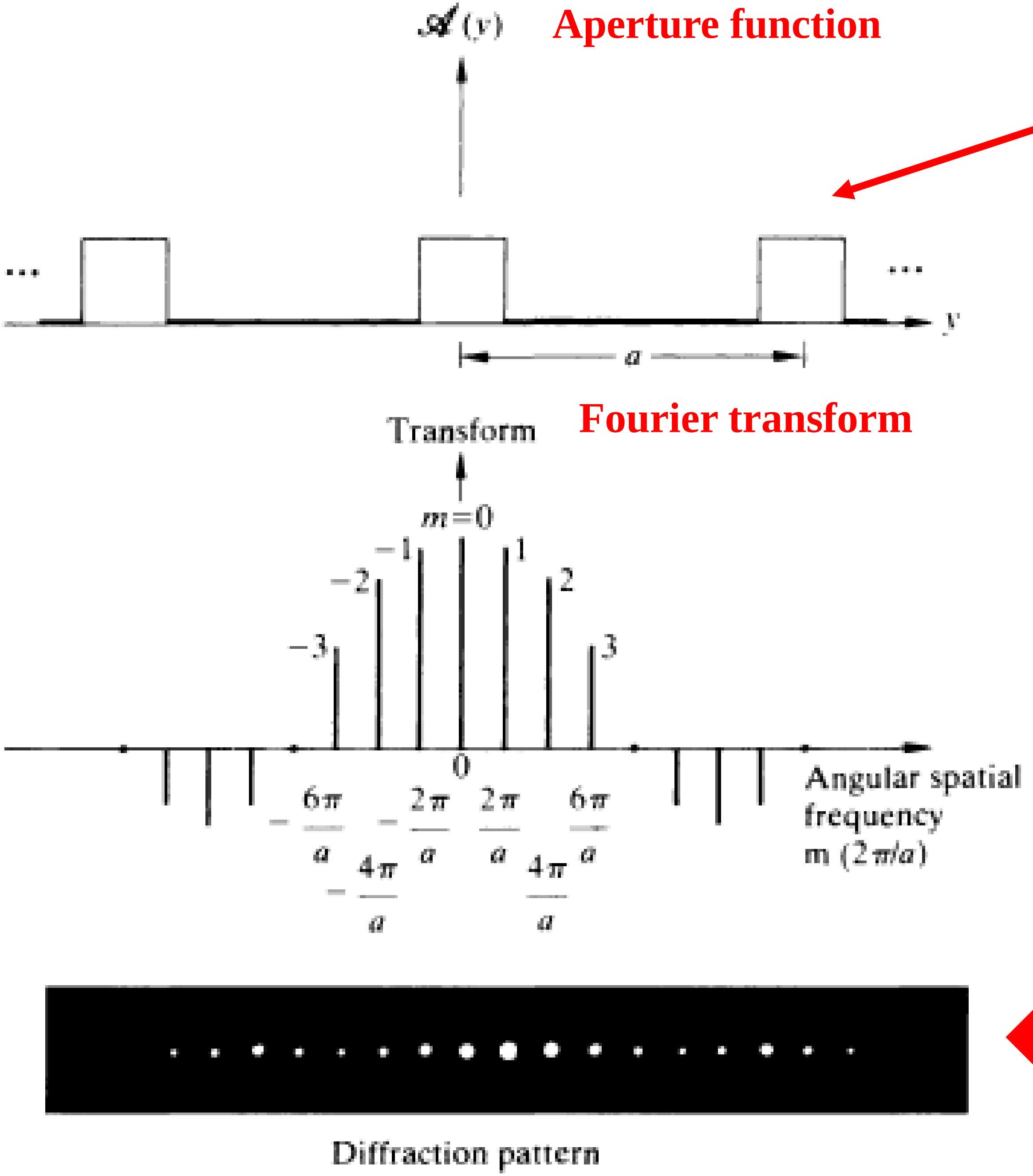
- Consider an object with periodic structure such as the **Ronchi ruling**, a grating of parallel straight lines with large grating space, whose opaque and transparent regions are of equal width.
- The object is illuminated from behind by a **monochromatic plane wave**.
- The Ronchi ruling acts as a coarse grating producing a series of bright spots that correspond to the various order of diffraction.

https://en.wikipedia.org/wiki/Ronchi_ruling

**Aperture function**

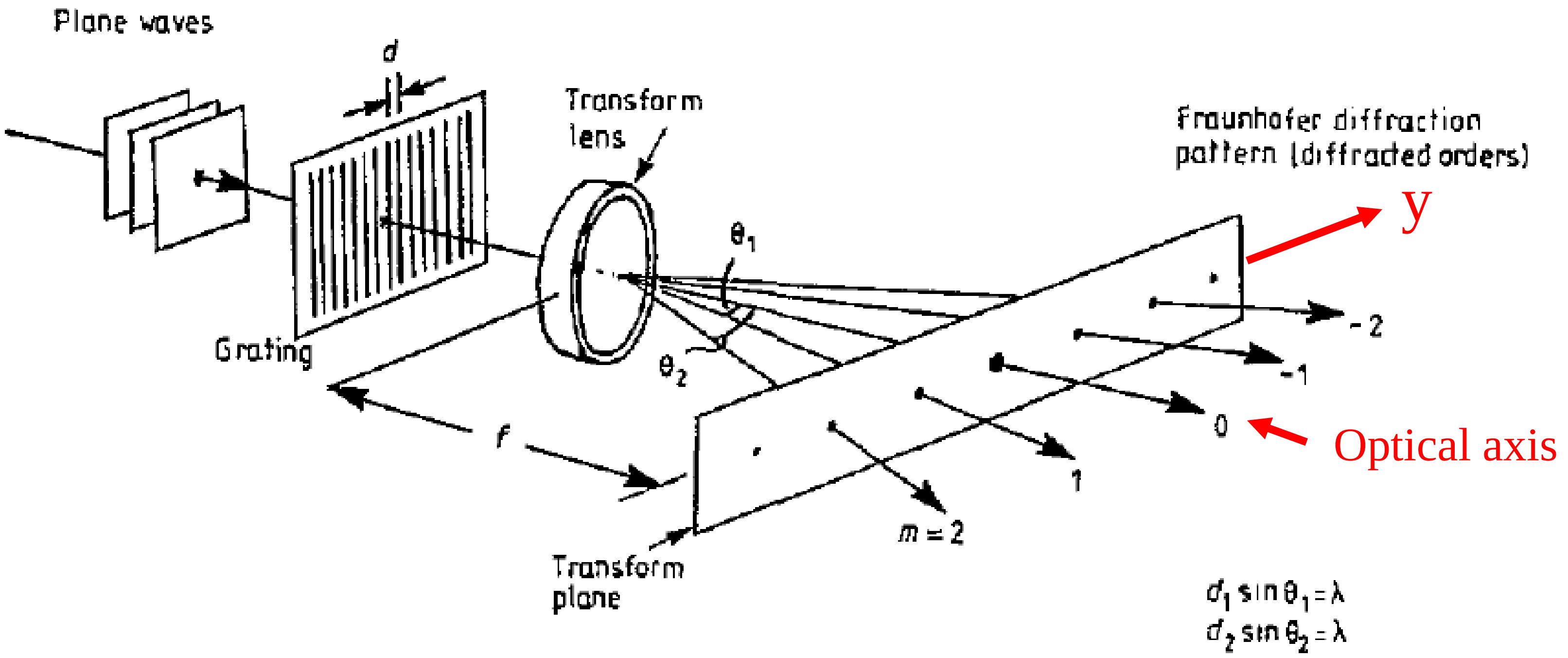
**The aperture function introduced by Ronchi ruling can be represented by a periodic step function**

**Fourier transform**




- Each spot of light in the diffraction pattern denotes the presence of a specific spatial frequency, which is proportional to its distance from the optical axis (zero-frequency) location.

**Diffraction pattern in terms of irradiance corresponding to Fourier transform**




- Suppose the spectrum of bright spots are aligned along the Y-axis.
- According to the grating equation, $m\lambda = d \sin\theta = d\frac{Y_m}{f}$ where *d* is the spatial period of the ruling and *f* is the focal length of the transform lens.
- Spots appear at distances $Y_m$ from the optical axis is given by $Y_m = m\left(\frac{\lambda f}{d}\right)$

# Revisit Fraunhofer diffraction

- Recall 2D Fourier transform

$$A_p(k_X, k_Y) = \iint E_A(x, y) e^{i(k_X x + k_Y y)} dx dy$$

- This shows that the **amplitude distribution** or **Fraunhofer diffraction pattern** $A_p\ (k_X, k_Y)$ actually is the **2D Fourier transform** of the **aperture function** $E_A\ (x,y)$.
- Also, recall the **angular spatial frequencies**, $k_X \equiv \frac{kX}{r_0}$ and $k_Y \equiv \frac{kY}{r_0}$
- Under this circumstance, the amplitude distribution is focused on y axis only and distance $r_0$ is replaced by the focal length $f$.
- The angular spatial frequency becomes $k_Y \equiv \frac{kY}{f}$
- By definition, the angular spatial frequency may be written as $k_Y \equiv 2\pi\nu_Y$ $\because Y_m = m\left(\frac{\lambda f}{d}\right)$
- This gives $\nu_Y = \frac{m}{d}$ corresponding to the **spectrum of spatial frequencies** displayed in the diffraction pattern.

# Spectrum of spatial frequencies : $\nu_Y = \frac{m}{d}$

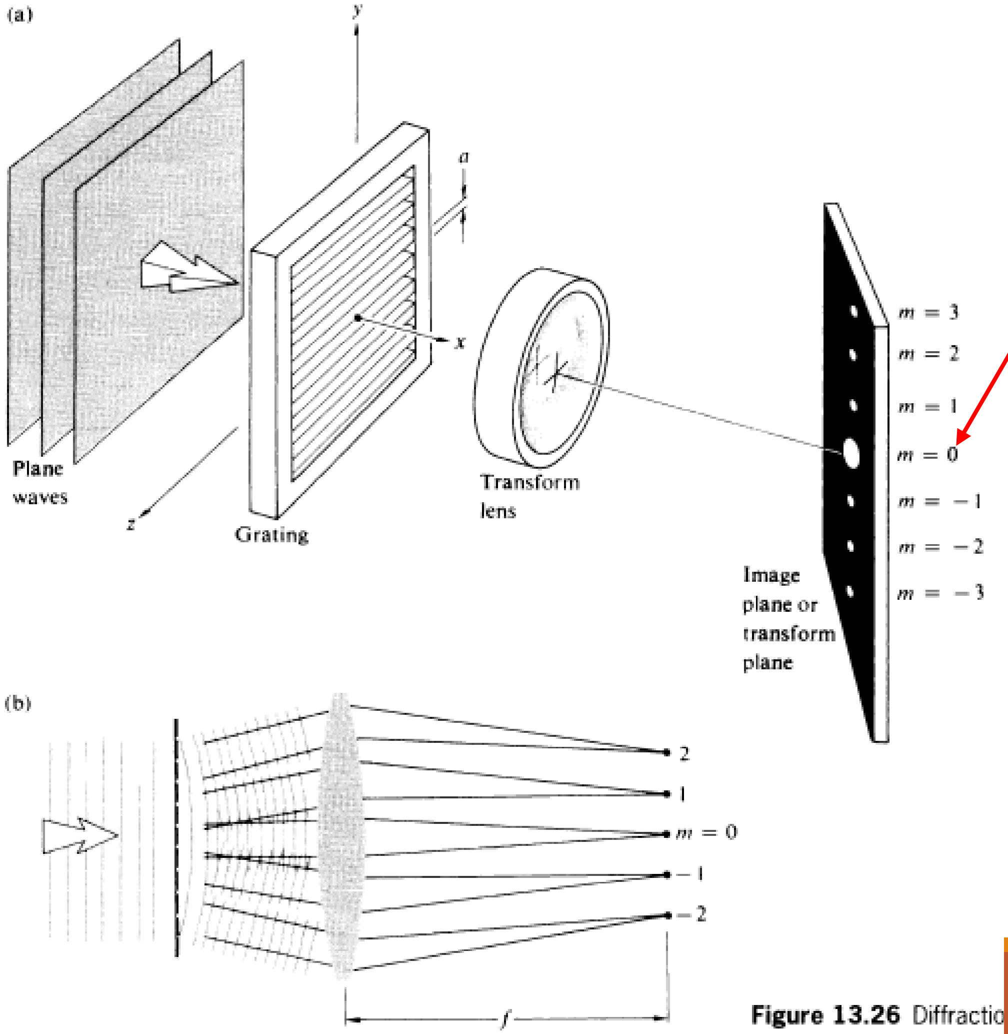


Figure 13.26 Diffractio

- The central spot with $m = 0$ corresponds to a normalized spatial frequency $\nu_Y = 0$, the DC component.
- The first order ($m = 1$) spots above and below the central spot represent the **fundamental spatial frequency $\nu_{Y1} = 1/d$**.
- Higher order ($m$>1) spots represent higher harmonics given by $m\,\nu_{Y1}$ .
- **Each spot of light in the diffraction pattern denotes the presence of a specific spatial frequency, which is proportional to its distance form the optical axis (zero-frequency location).**

# Problem 8

- Consider a Ronchi ruling with slits of width 0.2 mm illuminated by light of wavelength 488 nm. A lens of focal length 40 cm is used.

Determine

(a) the distances of the $m = 1$ and $m = 3$ spots from the central DC spot in the diffraction pattern on the screen in the spectrum plane.

(b) the angular spatial frequencies associated with the $m = 1$ and $m = 3$ spots.

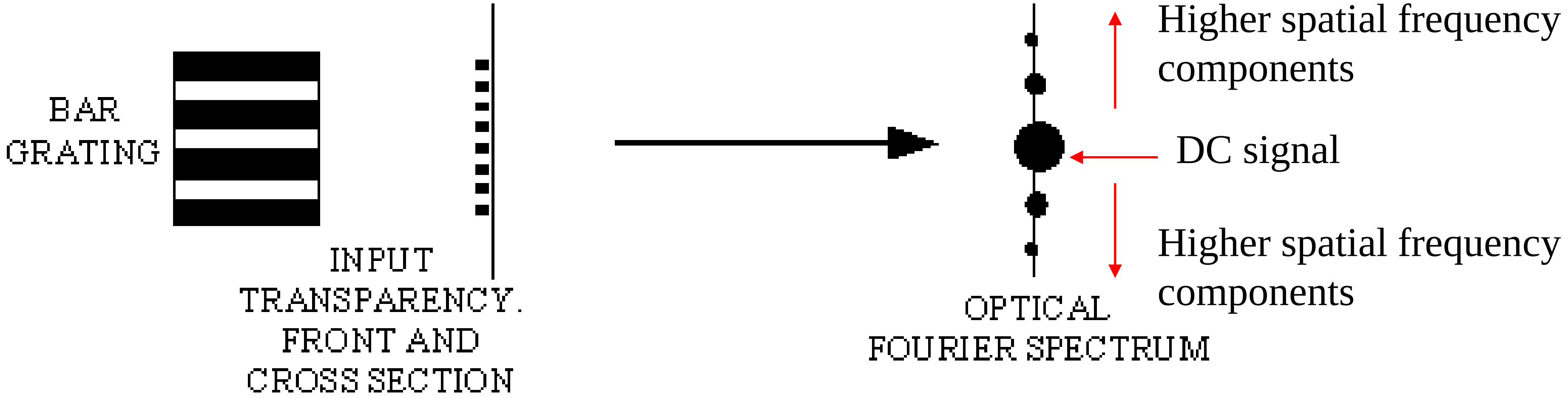
# Solution

- (a) Recall $Y_m = m\left(\frac{\lambda f}{d}\right)$ and replace each variable with an appropriate numerical value.

$$Y_1 = (1)\frac{(488\times10^{-9})(0.4)}{0.2\times10^{-3}} = 0.976\ \text{mm}, Y_2 = 2.93\ \text{mm}$$

- (b) By definition of an angular spatial frequency : $k_Y = 2\pi\nu_Y$ where $\nu_Y = \frac{m}{d}$

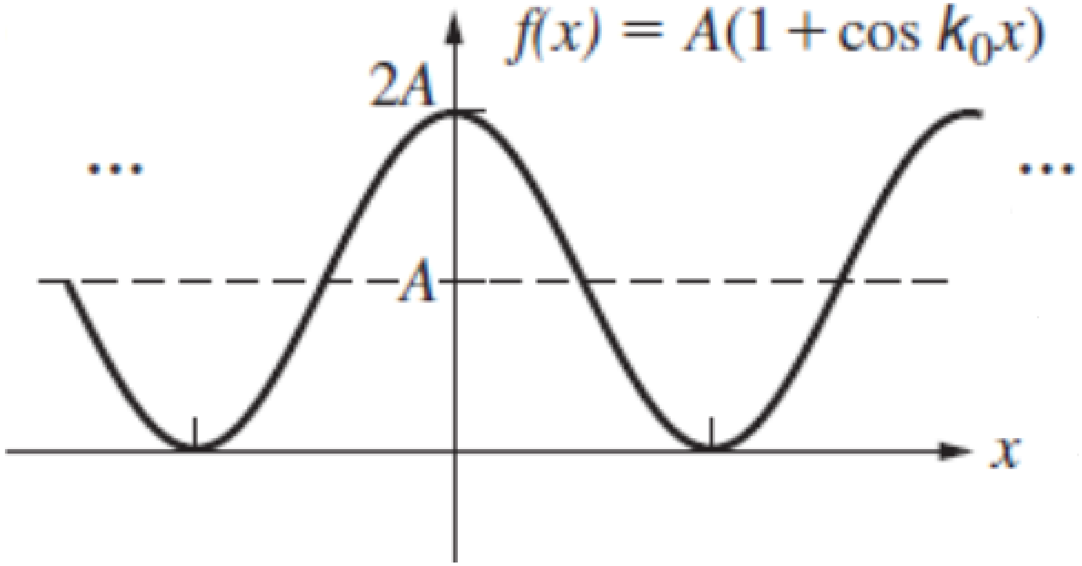
$$\therefore k_{Y1} = 2\pi\left(\frac{1}{0.2}\right) = \frac{31.4}{\text{mm}}, k_{Y2} = 2\pi\left(\frac{3}{0.2}\right) = \frac{94.2}{\text{mm}}$$




http://www.eng.warwick.ac.uk/~espbc/courses/undergrad/lec8/Image2.gif

# The DC contribution in the diffraction pattern

- **From the diffraction due to a grating, a DC spot (no spatial frequency) is always present.**
- This seems to make a contradiction between what is produced from the Fourier transform (theory) and the diffraction.
- To make the theory comply with the experimental result, the aperture function has to be modified accordingly.
- Consider the next problem!




# Problem 9

- Determine the Fourier transform of f(x).

# Solution

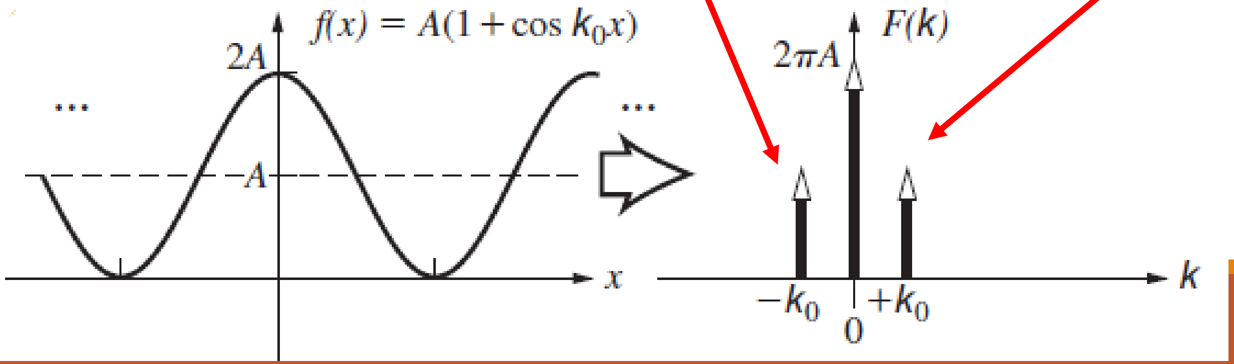
- $F(k) = F\{A + Acosk_0x\} = \int_{-\infty}^{\infty} Ae^{ikx}dx + \int_{-\infty}^{\infty} Acosk_0xe^{ikx}dx$
- From Problem 4, we already have $F\{A\} = 2\pi A\ \delta(k)$.
- Now we have to determine $F\{Acosk_0x\}$ which can rewritten as $\frac{A}{2}F\{e^{ik_0x} + e^{-ik_0x}\}$.
- Therefore, $F(k) = \int_{-\infty}^{\infty} A(cosk_0x)e^{ikx}dx = \frac{A}{2}\int_{-\infty}^{\infty} \{e^{ik_0x} + e^{-ik_0x}\}e^{ikx}dx$

$$= \frac{A}{2} \int_{-\infty}^{\infty} \{e^{i(k+k_0)x} + e^{i(k-k_0)x}\}\ dx$$

$$= \frac{A}{2}\{2\pi\delta(k + k_0) + 2\pi\delta(k - k_0)\}$$

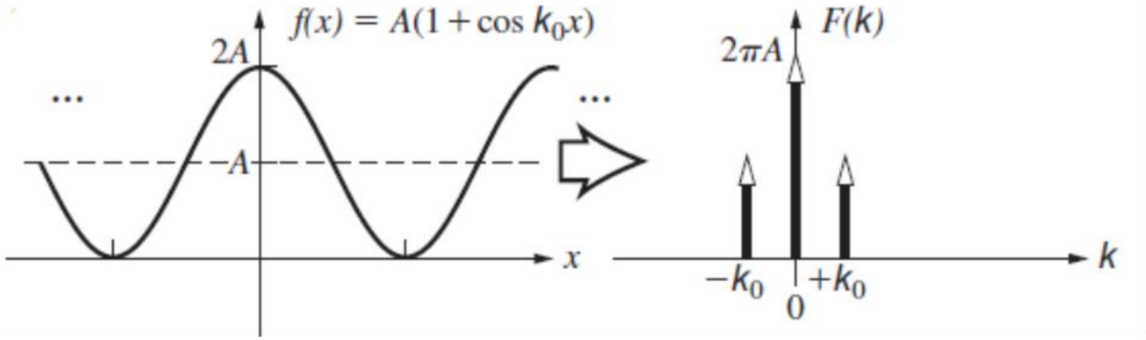
**Derive this!**

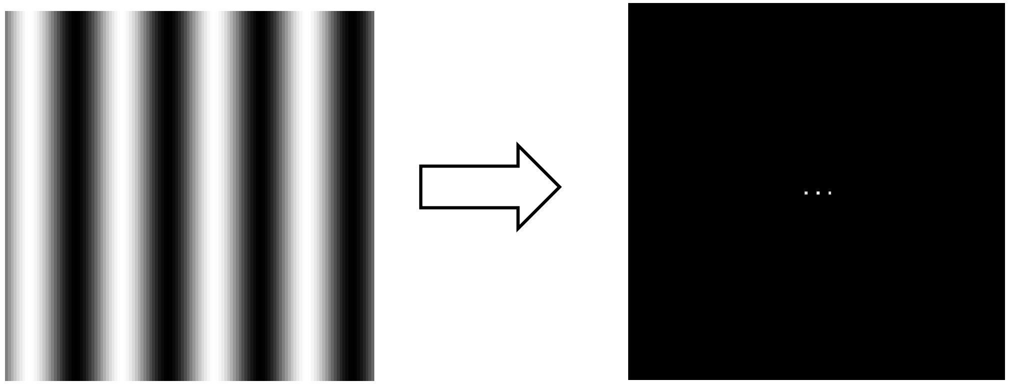
$$\therefore F(k) = F\{A + Acosk_0x\} = \pi A\delta(k + k_0) + 2\pi A\ \delta(k) + +\pi A\delta(k - k_0)$$



# Interpretation of the Fourier transform and its corresponding diffraction pattern

**Modified aperture function** **Fourier transform**



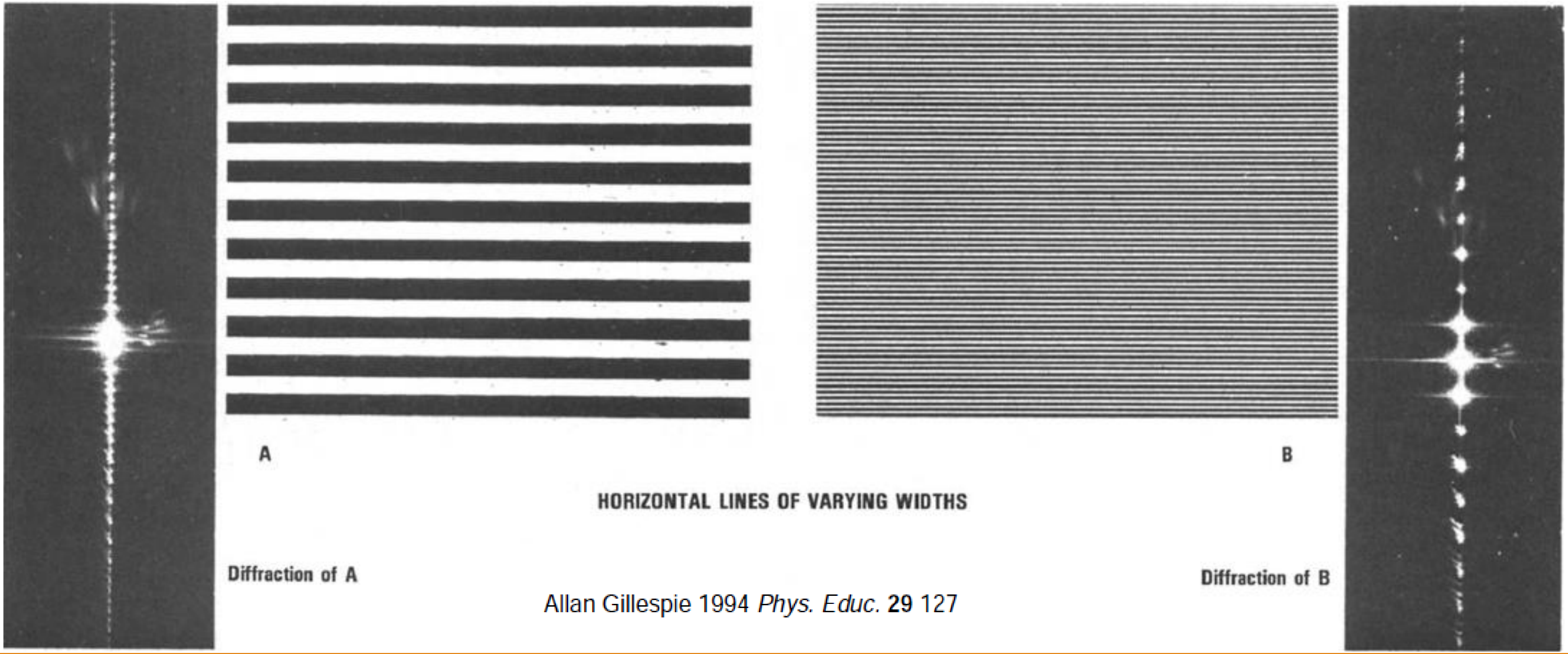



**Aperture** **Diffraction pattern**

- The modified aperture function gives rise to an addition DC component (at $k = 0$).
- The three spatial frequency components in the Fourier transform correspond to the three bright spots appearing in the diffraction pattern.
- Note that the DC contribution is thought to be originated from a uniform grey background and this must be present in all physical images of this sort.

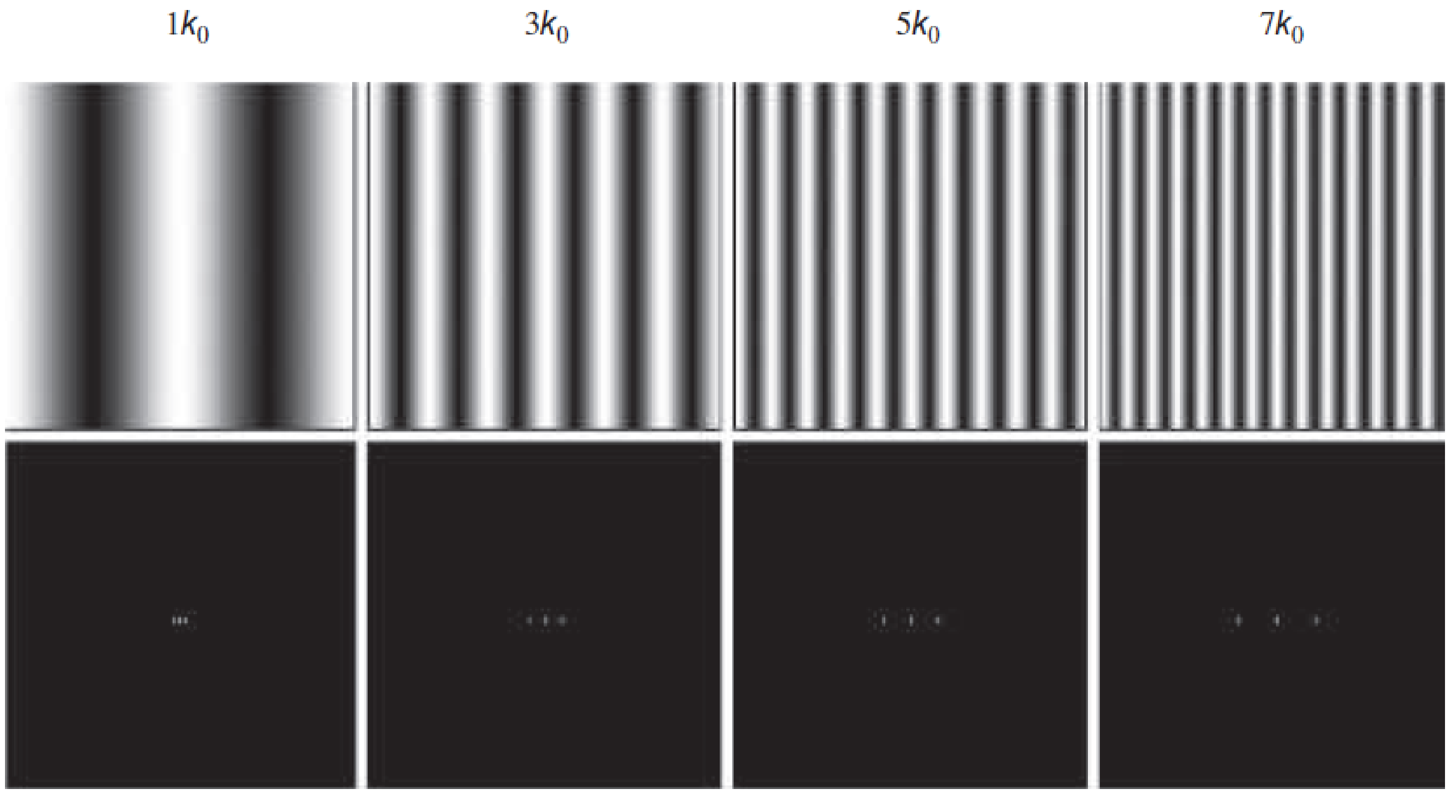
http://www.imagemagick.org/Usage/fourier/

# Diffraction pattern from horizontal lines of varying widths




Allan Gillespie 1994 *Phys. Educ.* **29** 127

# Diffraction patterns from sinusoidal gratings of varying spatial frequencies

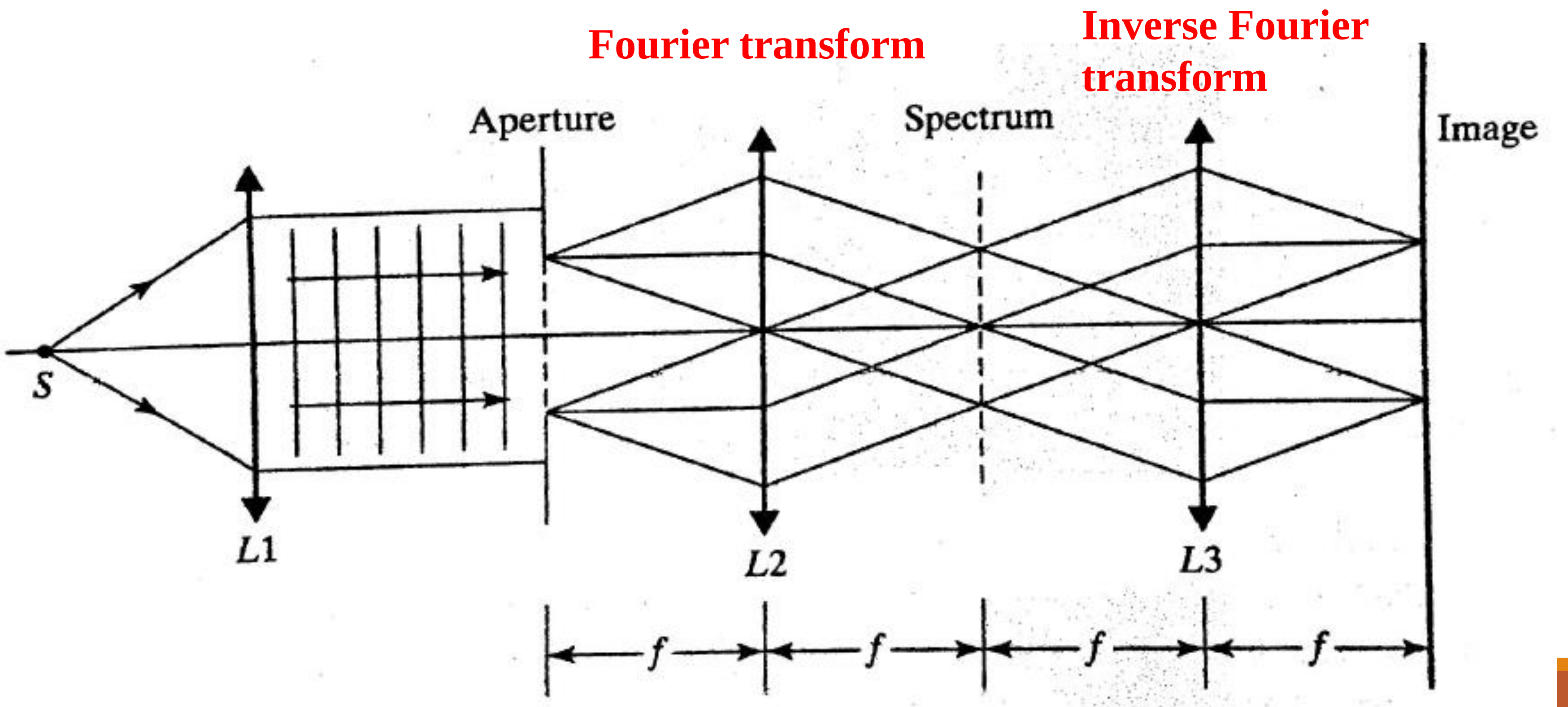



$k_0$ = fundamental spatial frequency

- Several brightness **sinusoidal signals** and their Fourier transforms.
- The spatial frequency ranges from that of the fundamental $k_0$ to the third, fifth, and seventh harmonics.

# Optical filtering : arrangement

- The concept of Fourier transform can be applied to the process of intentionally **blocking** certain portion- certain spatial frequencies- present in the diffraction pattern, to manipulate the image.
- First of all consider the arrangement of the optical filter.

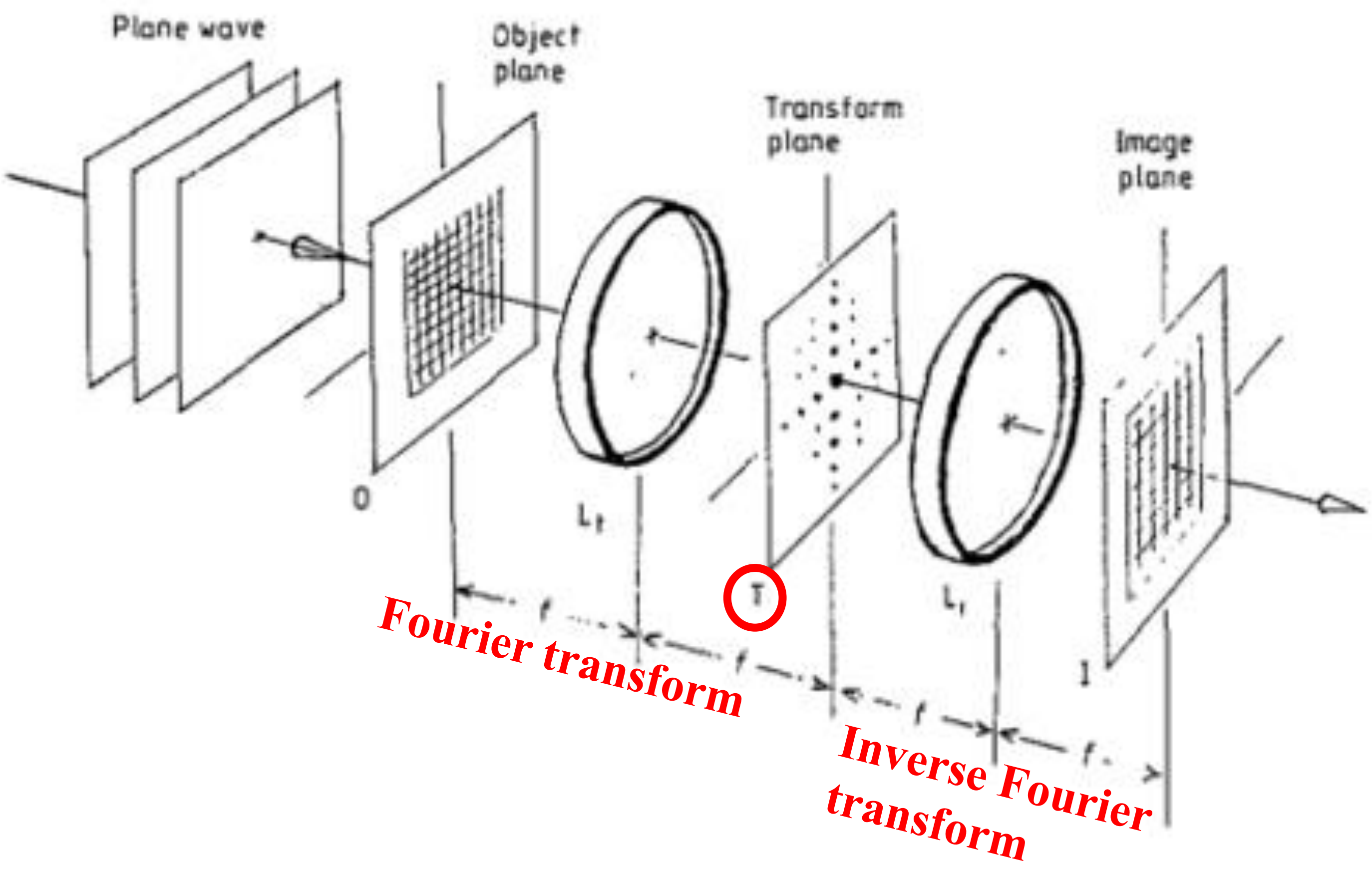



- Fourier transform of the aperture function is located at the focal plane (spectrum plane) of L2.
- The spectrum plane of L2 serves as an aperture function for L3 and associated Fourier transform which is the original aperture function is formed on the image plane

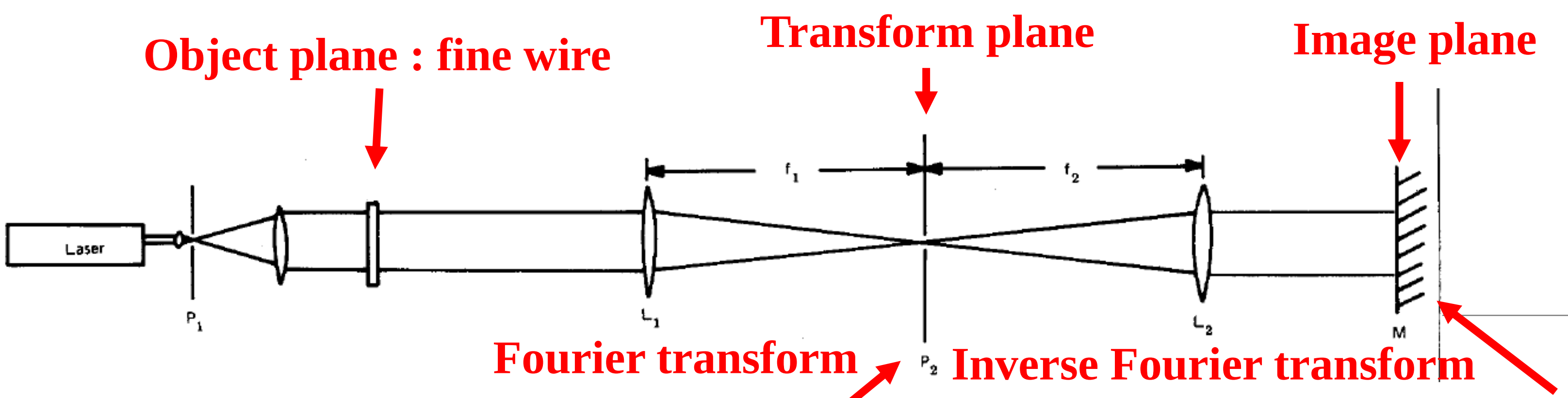
# Optical filtering : operation

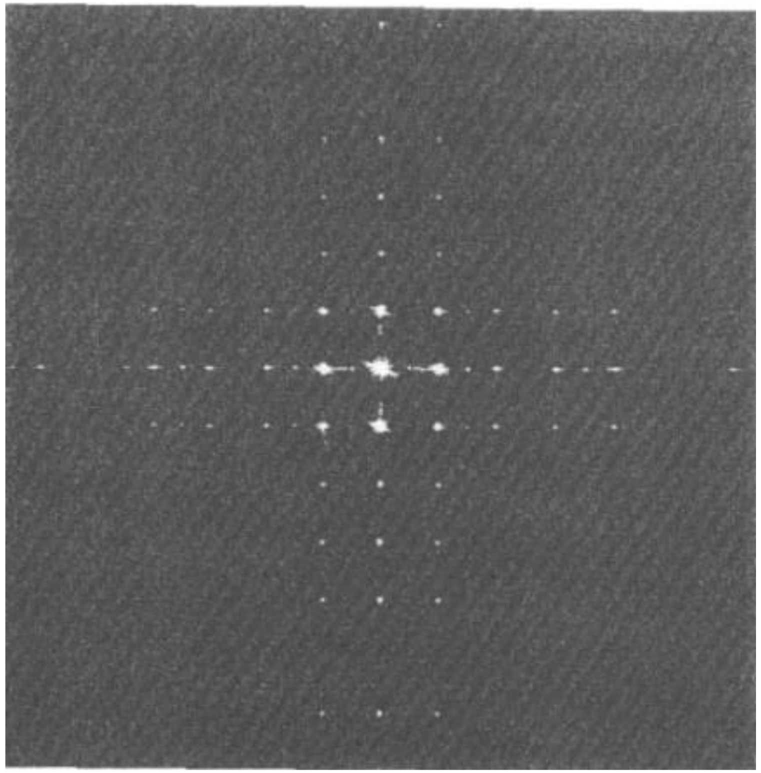
- Since the aperture function can be the superposition of noise such as periodic horizontal lines and a desired signal.
- When Fourier transform is applied to the **aperture function** which contains the noise and desired signal. The diffraction pattern due to  periodic **horizontal lines** can produce a series of diffraction spots along the **vertical direction** in the spectrum plane.
- If the diffraction spots can be blocked somehow, the periodic horizontal lines are filtered out and the final image is the reproduction of the desired signal without the periodic horizontal line present.

# 4-f coherent imaging system




- An object is illuminated by a plane wave emitted from a laser.
- Two identical lenses $T_t$ and $T_i$ perform transform and inverse transform, respectively.
- This system performs the spatial filtering by which certain spatial frequencies that make up and object are removed.
- This can be done by inserting an appropriate mask at the transform plane T.



Diffraction pattern of wire mesh on transform plane

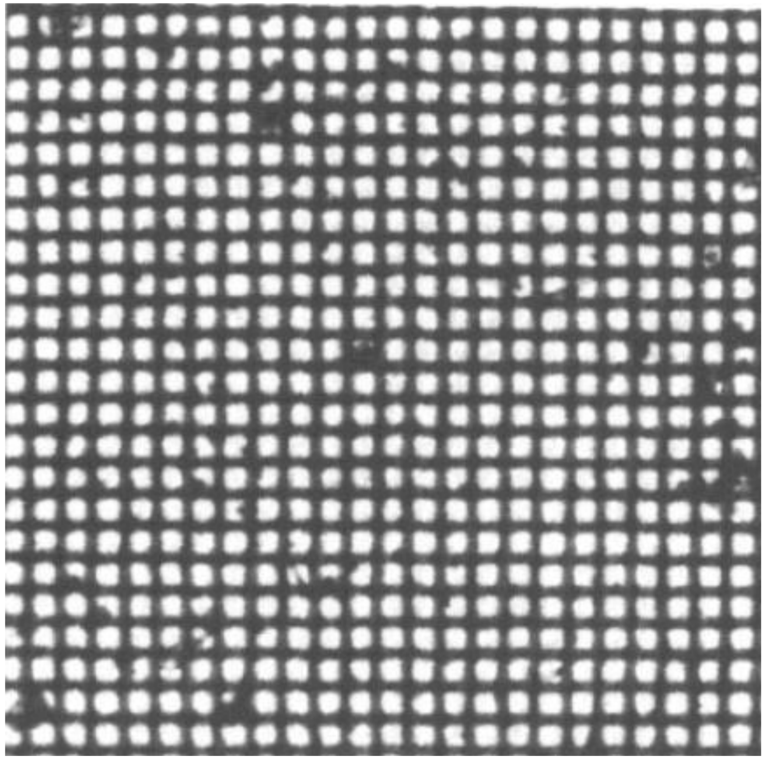
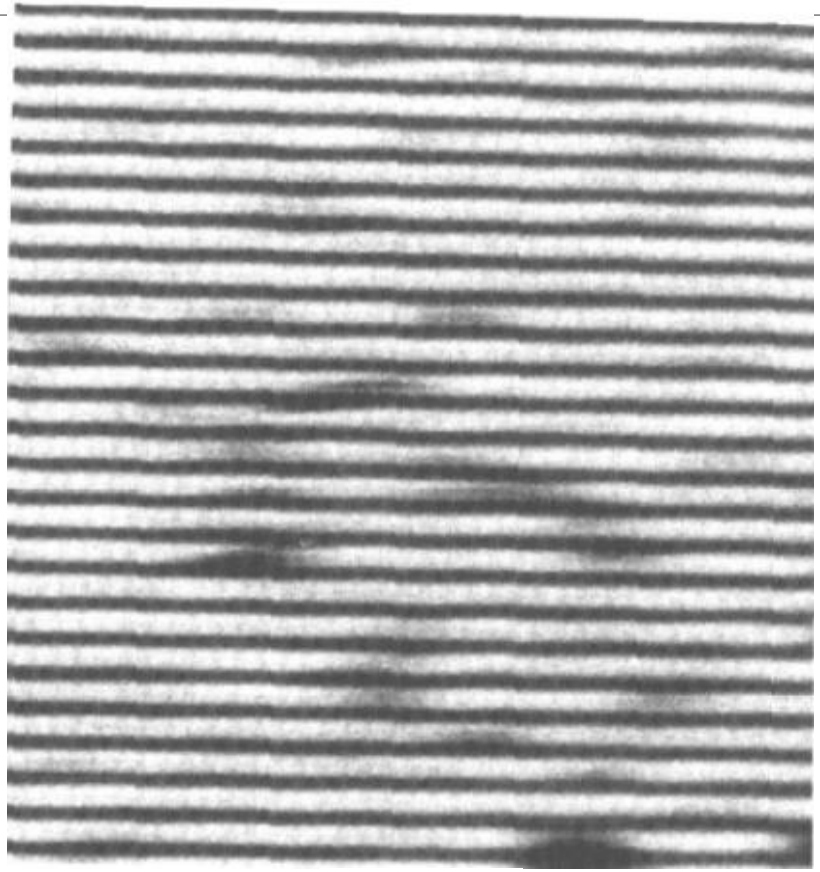
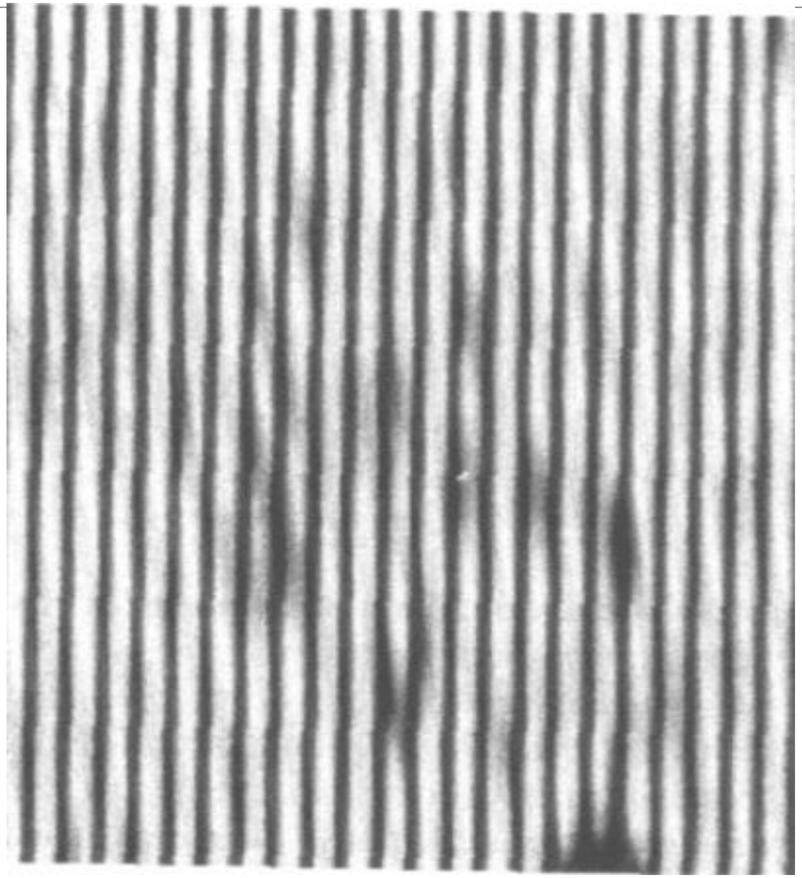


Image of fine wire seen on image plane without filter

**Following the set up in the previous slide.**
**Which image is produced by blocking high spatial frequency components in horizontal components?**

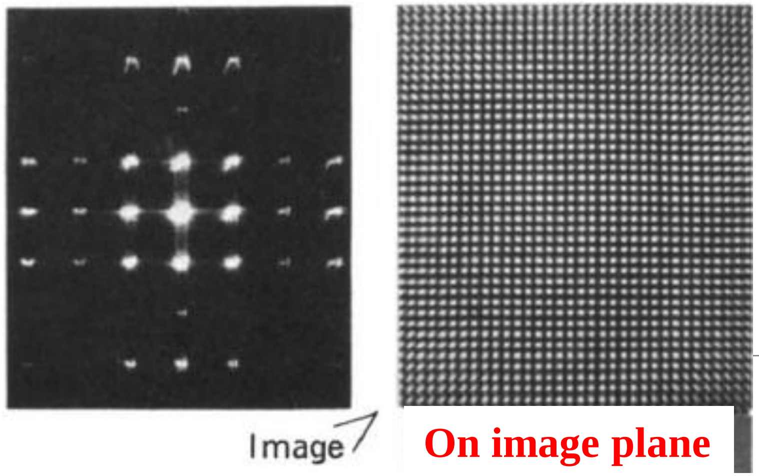




Different images are obtained when only certain parts of the diffraction pattern are permitted to contribute to this image. The other portions of the diffraction pattern are blocked by opaque tape.

No Filter (All of Diffraction Pattern Passes)

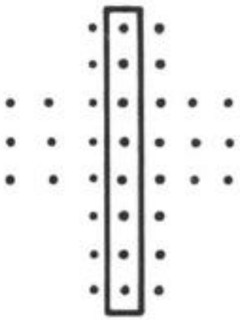
**On transform plane**




**On image plane**

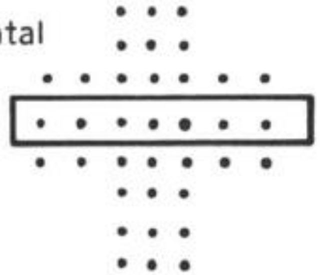
Vertical Dots

**permitted**



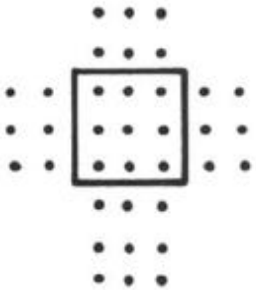
Horizontal Dots

**permitted**



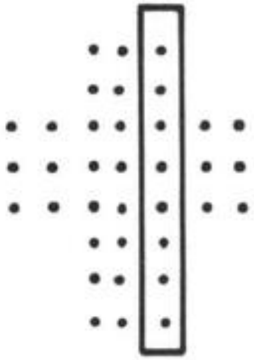
Central Nine Dots

**permitted**



Vertical Dots (Off Center)

**permitted**



# Improved lunar image  from Lunar orbiter

The image was obtained from the Lunar Orbiter. It is a panorama with evident bright lines between each stitched frame. We want to eliminate these lines to render a smooth image. Notice that these horizontal lines occur periodically in the image. Therefore, its Fourier transform is composed of points along the y-axis. By blocking the signal along the y-axis of its transform, the result is a smooth picture as shown in (c)

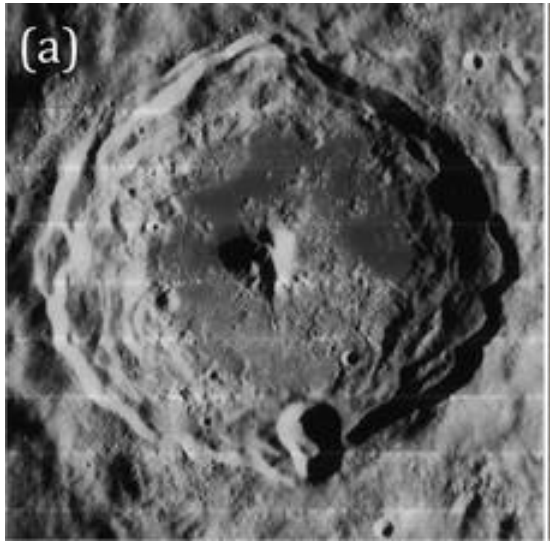
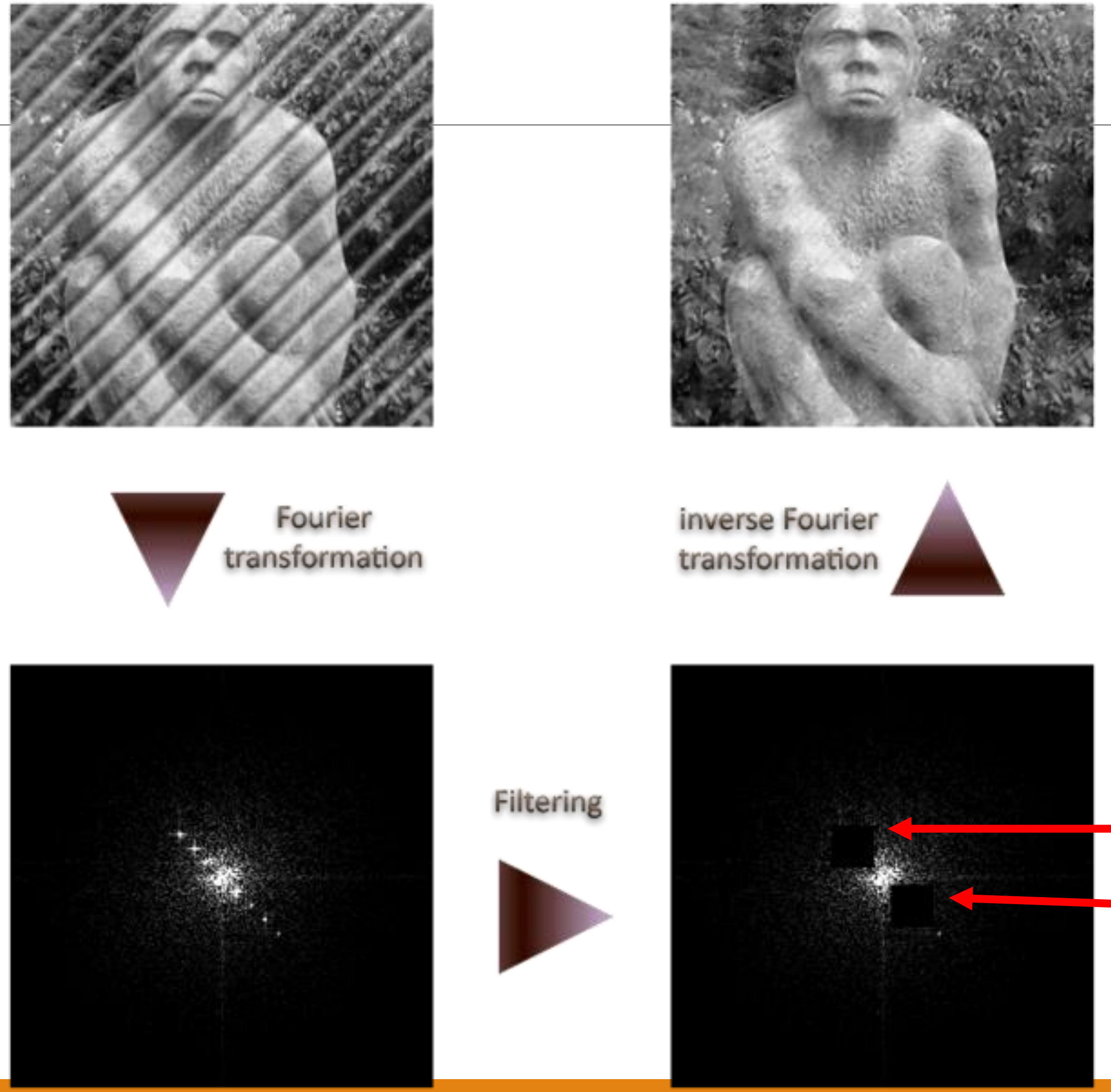



Image from Lunar orbiter (a), its filtered Fourier transform (b) and the rendered image after filtering (c).

https://ghdoblado.wordpress.com/2016/10/13/properties-and-applications-of-2d-fourier-transform/

# Defect removal in the spatial frequency domain using two-dimensional Fourier transform of an image



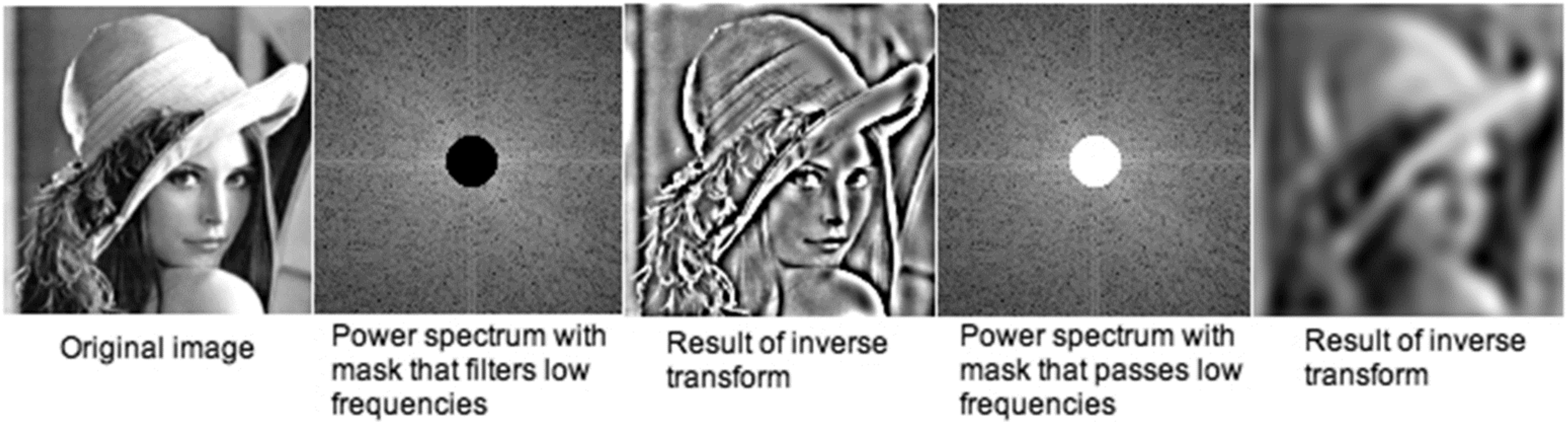

**Notice : the diffraction components caused by the tilted co-sinusoidal band are blocked.**

https://commons.wikimedia.org/wiki/File:Fourier_filtering.png

# How does an image appear if all spots are blocked except the DC component, or undeviated diffraction?

- From the knowledge of Fourier series, the finer features of the image disappear when spots corresponding to the higher spatial frequencies are blocked.
- A diaphragm can be used to blocked all spatial frequency components except the DC component.
- From the point of view of  optical filtering, the diaphragm, which *blocks all but frequencies near the direct beam*, functions as a **low-pass optical filter**.
- A diaphragm, which *blocks only those frequencies near the direct beam*, functions as a **high-pass optical filter**.
- A clear annular ring, which *blocks the lowest and the highest frequencies*, functions as a **band-pass optical filter.**

# Results of high and low spatial filtering




**High pass filter** **Low pass filter**

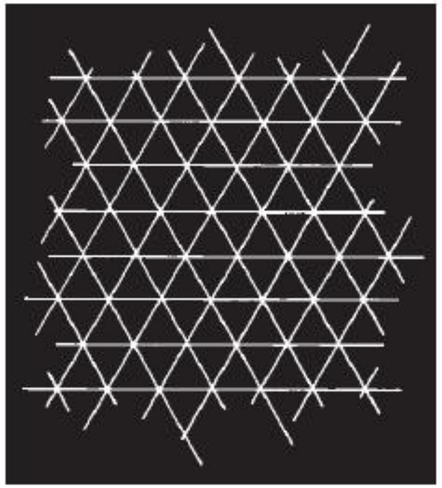
# Homework#12

**11.4*** Show that $\mathscr{F}\{1\} = 2\pi\delta(k)$.

**11.5*** Determine the Fourier transform of the function $f(x) = A\cos k_0 x$.

**13.39** What would the pattern look like for a laserbeam diffracted by the three crossed gratings of Fig. P.13.39?

**Figure P.13.39**



**13.40** Make a rough sketch of the Fraunhofer diffraction pattern that would arise if a transparency of Fig. P.13.40*a* served as the object. How would you filter it to get Fig. P.13.40*b*?

**Figure P.13.40** (E.H.)